Fourier Theory

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Konstantin Tretyakov

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1 Decomposition of functions into sinewaves

1.1 Why sinewaves?

Sinewaves are ubiquitous in sound analysis—a sinewave is considered to be the "pure sound of a given frequency" and it is customary to regard the more complex soundwaves as sums of sines or cosines. There are at least two good reasons for that:

- **Physical/physiological**. A sinewave describes the motion of an ideal harmonic oscillator. Most of the music instruments as well as human vocal cords can be regarded as harmonic oscillators. Also, the basilar membrane in the human ear is, in a sense, decomposing incoming signal to sinewaves by *resonating* with different frequency at different points. In short, we generate and perceive sound as a combination of sinewaves. It is of no surprise then, that the functions $f(t) = \sin(440 \times 2\pi t) + \sin(660 \times 2\pi t)$ and $g(t) = \sin(440 \times 2\pi t) + \cos(660 \times 2\pi t)$, despite looking differently, are perceived as approximately the same sound.
- **Mathematical**. A sinewave turns out to be the *eigensignal* of a *linear time-invariant transformation*. Linear time-invariant transformations correspond precisely to what is known as *sound filters*. Linearity implies that increasing the input of the filter will only increase the output proportionally (i.e. the filter will work similarly on both quiet and loud sounds) and time-invariance means the filter does not care when the sound comes—right now, in a minute, or in an hour—it will be transformed equally. Finally, the fact that sinewave is an *eigensignal* simply means that if you input a sinewave to a linear filter, you'll always get a (potentially scaled or shifted) sinewave on the output. This means that if you could represent a given function f as a sum of sinewaves, application of linear filters would become extremely simple—you'd only need to rescale the coefficients and shift the phases. This is often expressed by saying that *convolution in the signal space corresponds to product in the Fourier space*, we'll get to that later.

1.2 Decomposition, how's that?

As it should be clear from the previous, we'll be dealing with methods of representing a given function $f(t)$ as a weighted sum of sinewaves. Something like that, for example:

$$f(t) = 2\sin(t) + 3\sin(2t) - \sin(1.3t) \tag{1}$$

The concept should be known already from linear algebra, where a *vector* $\mathbf{v} \in \mathbb{R}^n$ could be represented in a given *basis* $\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$ as a *linear combination*:

$$\mathbf{v} = \alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2 + \dots + \alpha_n\mathbf{b}_n. \tag{2}$$

Note that a vector can be regarded as a function $\mathbf{v} : \mathbb{N} \to \mathbb{R}$. Conversely, a function can be imagined as a vector with an infinite number of components. Hence, the analogy between (1) and (2) is quite strong.

We know from linear algebra that in an n-dimensional vector space there exist different *bases*, each consisting of exactly n elements, such that *every vector* from the space can be represented as a linear combination of the basis vectors. Moreover, if we introduce the notion of an *inner product* $\langle \cdot, \cdot \rangle$ between vectors, we can always select an *orthonormal basis*, which is particularly convenient for representing vectors. If $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is orthonormal, then for any $\mathbf{v} \in \mathbb{R}^n$, the coefficients of (2) can be found as:

$$\alpha_i = \langle \mathbf{v}, \mathbf{b}_i \rangle. \tag{3}$$

As you'll see, it's all very similar for functions, however slightly more complicated. First of all, the space of functions is *infinite dimensional*, hence it's clear that a finite basis won't suffice. Therefore we'll have to select an *infinite* set of basis functions $\{b_i(t)\}_i$, and attempt to represent a given function $f(t)$ as a *series*:

$$f(t) = \sum_{i=0}^{\infty} \alpha_i b_i(t). \tag{4}$$

Alternatively, we might also attempt to select an *infinite uncountable* set of basis functions $\{b_w(t)\}_w$ and attempt to represent $f(t)$ as an *integral*:

$$f(t) = \int_{w=0}^{\infty} \alpha(w) b_w(t) \mathrm{d}w \tag{5}$$

It still remains a question, whether there really exists a reasonable basis that would allow to represent most useful functions, and, if it exists, how to find the corresponding coefficients. For a positive example recollect the Taylor series. It holds that any *analytic function* $f(x)$ can be represented as a series of the form

$$f(x) = \sum_{i=0}^{n} \alpha_n x^n,$$

which is an example of representation (4) in the set of basis functions $\{x^i\}_i$.

2 Fourier series

2.1 The trigonometric basis

Now, let's select the family of sines and cosines as the basis, ie. $\{\sin(nt), \cos(nt)\}_n$ and examine what functions can we represent in it. In particular, we'll be interested in the representation of the form:

$$f(\phi) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt)). \tag{6}$$

The factor $\frac{1}{2}$ is there just to unify some of the equations which come further on. Equation (6) represents an (infinite) linear combination of functions $\frac{1}{2}$, $\sin(nt)$, $\cos(nt)$. As all of the basis functions are 2π-periodic and smooth, the resulting function must also be periodic and without "too many" discontinuities—so that we are far from being able to represent *any* function. However, we're examining soundwaves, which are periodic and mostly continuous anyway.

Let us now define the inner product on the space of integrable 2π-periodic functions as

$$\langle f(t), g(t)\rangle = \int_0^{2\pi} f(t)g(t)\mathrm{d}t.$$

It turns out that according to this inner product our chosen set of basis functions is *pairwise orthogonal*:

$$\langle\cos(mt), \sin(nt)\rangle = \int_0^{2\pi} \cos(mt)\sin(nt)\mathrm{d}t = 0$$

$$\langle\cos(mt), \cos(nt)\rangle = \int_0^{2\pi} \cos(mt)\cos(nt)\mathrm{d}t = \begin{cases} 2\pi & \text{if } m = n = 0 \\ \pi & \text{if } m = n > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\langle\sin(mt), \sin(nt)\rangle = \int_0^{2\pi} \sin(mt)\sin(nt)\mathrm{d}t = \begin{cases} \pi & \text{if } m = n > 0 \\ 0 & \text{otherwise} \end{cases}$$

This means that we'll be able to use the analogue of the equation (3) to compute the coefficients. Indeed, assume that $f(t)$ is of the form (6), and the series converges uniformly. Then, for $m > 0$:

$$\begin{aligned} \langle f(t), \cos(mt)\rangle &= \int_0^{2\pi} f(t)\cos(mt)\mathrm{d} \\ &= \int_0^{2\pi} \left(\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos(nt) + b_n\cos(nt))\right)\cos(mt)\mathrm{d}t \\ = \frac{1}{2}a_n \int_0^{2\pi} \cos(mt)\mathrm{d}t + \sum_0^{2\pi} &\left(a_n \int_0^{2\pi} \cos(nt)\cos(mt)\mathrm{d}t + b_n \int_0^{2\pi} \sin(nt)\cos(mt)\mathrm{d}t\right) \\ &= \pi a_m. \end{aligned} \tag{7}$$

Thanks to the constant $\frac{1}{2}$ in front of a_0, the equation also holds for $m = 0$. It follows then that

$$a_m = \frac{1}{\pi}\langle f(t), \cos(mt)\rangle = \int_0^{2\pi} f(t)\cos(mt)\mathrm{d}t. \tag{8}$$

Analogously,

$$b_m = \frac{1}{\pi}\langle f(t), \sin(mt)\rangle = \int_0^{2\pi} f(t)\sin(mt)\mathrm{d}t. \tag{9}$$

The functions a_m and b_m defined by equations (8) and (9) are known as the *Fourier coefficients* of $f(t)$. Of course, not every function $f(t)$ can be represented as the series (6), and of those that can, not every one satisfies the uniform convergence requirement. Luckily enough, for most "nice" functions the representation (6) exists and the Fourier equations hold.

It can be shown, that Fourier representation holds for any $f(t)$, that has a *bounded continuous derivative except at a finite number of points*[1] *in* $[0, 2\pi]$. The series (6) then converges to $f(t)$ at each point where $f(t)$ is continuous. At each point a of discontinuity, the series converges to $\frac{f(a^+)+f(a^-)}{2}$. Convergence is always non-uniform near the discontinuity points—an effect known as the *Gibbs phenomenon*.

[1]There is also a trick, known as *Cesàro summation*, which allows to recover *any Riemann-integrable* $f(t)$ from its Fourier coefficients even if it does not satisfy the above conditions (Fejér's theorem).

Clearly, the integration need not necessarily be done from 0 to 2π, sometimes it's more convenient to integrate from $-\pi$ to π. Also, the equations can be generalized for a function of period T rather than 2π by rescaling the argument:

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(2\pi n\nu t) + b_n \sin(2\pi n\nu t))$$

$$a_m = 2\nu \int_0^T f(t)\cos(2\pi m\nu t)\mathrm{d}t$$

$$b_m = 2\nu \int_0^T f(t)\sin(2\pi m\nu t)\mathrm{d}t$$

where $\nu = \frac{1}{T}$ is the *fundamental frequency* of $f(t)$.

There are lots of things one can discover about the Fourier series for particular kinds of functions. For example, the series for odd functions contains sines only, the series for even functions—only cosines.

Fourier series can have different forms. For example, by using the identity $\cos(A-B) = \cos(A)\cos(B)+\sin(A)\sin(B)$ it is possible to rewrite each term $a_n\cos(nt)+b_n\sin(nt)$ in the form $\alpha_n\cos(nt+\phi_n)$. However, one of the most convenient forms is the *complex exponential series*.

2.2 Complex Fourier series

With the help of the Euler's equation $e^{it} = \cos(t) + i\sin(t)$ we can rewrite (6) as

$$f(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{int}, \tag{10}$$

where $\alpha_0 = \frac{1}{2}a_0$, and for $m > 0$, $\alpha_m = \frac{1}{2}a_m + \frac{1}{2i}b_m$, $\alpha_{-m} = \frac{1}{2}a_m - \frac{1}{2i}b_m$.

We could also derive the equations for α_i by introducing the inner product

$$\langle f(t), g(t)\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}\mathrm{d}t,$$

and noticing that the basis functions $\{e^{int}\}_n$ are pairwise orthonormal, hence

$$\alpha_m = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-imt}\mathrm{d}t. \tag{11}$$

For the T-periodic function $f(t)$, we just have to rescale the argument:

$$f(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{2\pi in\frac{t}{T}} \qquad \alpha_m = \frac{1}{T}\int_0^T f(t)e^{-2\pi im\frac{t}{T}}\mathrm{d}t. \tag{12}$$

3 The Fourier transform

3.1 Fourier transform and inverse Fourier transform

The Fourier series representation can only handle periodic functions. Intuitively, one reason for that is in the fact that a *countable* set of sinewave-like functions is not "powerful" enough to represent a non-periodic function. Will taking an *uncountable* set and employing the integral representation (5) help? Indeed it will. It turns out that the basis $\{e^{2\pi i\nu t}\}_{\nu\in\mathbb{R}}$

is, in a sense, orthonormal with respect to the inner product $\langle f(t), g(t)\rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)}\mathrm{d}t$, so analogously to the previous cases we obtain:

$$f(t) = \int_{-\infty}^{\infty} F(\nu)e^{2\pi i\nu t}\mathrm{d}\nu, \tag{13}$$

$$F(\nu) = \int_{-\infty}^{\infty} f(t)e^{-2\pi i\nu t}\mathrm{d}t. \tag{14}$$

Equation (13) is known as the *inverse Fourier transform* of $F(\nu)$, and equation (14) — as the *Fourier transform* of $f(t)$.

It is common to illustrate Fourier transform as the limiting form of the complex Fourier series for a T-periodic function as T tends to infinity. It goes as follows: Let $f(t)$ be T-periodic. Then, using (12) we obtain:

$$f(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{2\pi i \frac{n}{T} t} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-2\pi i \frac{n}{T} t}\mathrm{d}t\right) e^{2\pi i \frac{n}{T} t}. \tag{15}$$

The equation is the Riemann sum of the kind $\sum \phi_{t,T}(\nu_n)\Delta\nu$, where $\nu_n = \frac{n}{T}$, $\Delta\nu = \frac{1}{T}$ and

$$\phi_{t,T}(\nu) = \left(\int_{-T/2}^{T/2} f(t)e^{-2\pi i\nu t}\mathrm{d}t\right) e^{2\pi i\nu t}.$$

Therefore, as T tends to infinity, $\phi_{t,T}(\nu)$ tends to $\phi_{t,\infty}(\nu)$, $\Delta\nu$ tends to 0 and the whole sum (15) converges, if anywhere, to the Riemann integral:

$$f(t) = \int_{-\infty}^{\infty} \phi_{t,\infty}(\nu)d\nu = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t)e^{-2\pi i\nu t}\mathrm{d}t\right) e^{2\pi i\nu t}\mathrm{d}\nu,$$

which demonstrates the Fourier transform and the inverse Fourier transform simultaneously[2].

In order for the Fourier transform to make sense, the function $f(t)$ must satisfy the same conditions that allowed a periodic function to be expanded in the Fourier series. In addition, for the integral to have a finite value, the function must be *absolutely integrable*, i.e. $\int_{-\infty}^{\infty} |f(t)|\mathrm{d}t < \infty$. However, the latter requirement can be relaxed with the help of the apparatus of *generalized functions (distributions)*.

The trigonometric and complex Fourier series and the Fourier transform are of course tightly related. As a simple example consider a function $f(t)$, which is nonzero only on the segment $\left[-\frac{T}{2}, \frac{T}{2}\right]$. Let there exist its Fourier transform $F(\nu)$. Now let $\hat{f}(t)$ be a T-periodic function, created by repeating $f(t)$, and let α_m be the complex Fourier coefficients of $\hat{f}(t)$. It's easy to confirm now, by comparing (12) and (14), that

$$\alpha_m = \frac{1}{T}F\left(\frac{m}{T}\right).$$

As the Fourier transform is more general than the trigonometric or the complex series, it is usually preferred to the latter forms and used much more often in practice.

[2]Note, that both the linear-algebra-based explanation, as well as the limit-of-the-series one are given here in a rather frivolous and sketchy manner, omitting a lot of technicalities related to the fact that you need to make and check several assumptions when switching integrals, sums and limit operators.

3.2 Examples

Box function

$$f(t) = \begin{cases} 1 & \text{if } t \in \left[-\frac{1}{2}, \frac{1}{2}\right] \\ 0 & \text{otherwise} \end{cases},$$

$$F(\nu) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i\nu t}\mathrm{d}t = \left.\frac{e^{-2\pi i\nu t}}{-2\pi i\nu}\right|_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{1}{\pi\nu}\sin\pi\nu = \mathrm{sinc}(\pi\nu).$$

Gaussian

$$f(t) = e^{-\pi t^2},$$
$$F(\nu) = e^{-\pi\nu^2}.$$

Cosine

$$f(t) = \cos(2\pi\nu_0 t) = \frac{e^{2\pi i\nu_0 t} + e^{-2\pi i\nu_0 t}}{2}$$
$$F(\nu) = \frac{\delta(\nu-\nu_0) + \delta(\nu+\nu_0)}{2}$$

Here $\delta(t)$ denotes the *Dirac delta function.* Albeit not really a function, it can be imagined as such. Namely,

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}, \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)\mathrm{d}t = 1$$

More formally, $\delta(t)$ is called a *distribution* and is defined via the linear operator $\int f(t)\delta(t)\mathrm{d}t$, acting on the space of integrable functions. By definition,

$$\int_{-\infty}^{\infty} f(t)\delta(t)\mathrm{d}t = f(0).$$

Although you can meaningfully add distributions and multiply them by other functions, they only really make sense within an integral. And, for example, you can take the Fourier transform of the delta function (which is one of the simplest Fourier transforms to take):

Dirac's delta function

$$f(t) = \delta(t) \qquad F(t) = \int_{-\infty}^{\infty} \delta(t)e^{-2\pi i\nu t}\mathrm{d}t = e^{-2\pi i\nu 0} = 1$$

Finally, note that the Fourier transform is in general a *complex-valued function.* The only reason why it turned out to be real-valued in the examples above is that these were all *even real-valued* functions.

It means that in general it's not enough to plot just one graph to visualize the Fourier transform. Two separate graphs are required, either for the real and the imaginary components, or for the amplitude and the phase of the spectrum. The latter possibility is more intuitive and is often the preferred one. The phase is sometimes considered as less important and not shown.

4 Properties of the Fourier transform

Despite the seemingly complicated integration, the procedure of taking Fourier transforms by hand can often (at least in the textbook examples) be reasonably simple, reminding that of differentiation, where you don't really compute the $\lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x}$, but rather use the well-known properties of the derivative to reduce the task to basic cases. Note that differentiation and Fourier transform are also similar in a sense that both are linear transformations on the function space. Differentiation is, in fact, *also* about a "change in the basis".

There are lots of useful properties of the Fourier transform, here are some of them (we use the notation $f(t) \sim F(\nu)$ to show the function and its Fourier transform):

$$\alpha f(t) + g(t) \sim \alpha F(\nu) + G(\nu) \tag{16}$$

$$F(t) \sim f(-\nu) \tag{17}$$

$$f(at) \sim \frac{1}{|a|} F\left(\frac{\nu}{a}\right) \tag{18}$$

$$f(t - t_0) \sim e^{-2\pi i \nu t_0} F(\nu) \tag{19}$$

$$f(t) e^{2\pi i \nu_0 t} \sim F(\nu - \nu_0) \tag{20}$$

$$f(t) * g(t) \sim F(\nu) G(\nu) \tag{21}$$

$$f'(t) \sim 2\pi \nu F(\nu) \tag{22}$$

$$\int_{-\infty}^{\infty} |f(t)|^2 \mathrm{d}t = \int_{-\infty}^{\infty} |F(\nu)|^2 \mathrm{d}nu \tag{23}$$

Equation (16) here shows the linearity of the Fourier transform; (17) says that the Fourier transform of a Fourier transform is the time-inversed original function (indeed, the equations of the forward and inverse transform are nearly identical). Equation (18) shows that a shorter signal will have a wider spectrum and vice-versa—a particular instantiation of the *uncertainty principle*. Equation (19) states that shifting the signal corresponds to the rotation of phases of each frequency component and (20) says that *modulating* the given signal with a higher-frequency wave shifts the spectrum of the signal upwards. Equation (21) demonstrates one particularly important property that was noted in the beginning. Namely, the spectrum of the *convolution* of two functions

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(s) g(t - s) \mathrm{d}s \tag{24}$$

is the *product* of their spectra. This property provides useful insight into the work of linear filters and allows to compute the filtering operation effectively. Besides, statisticians need it to prove the central limit theorem and more. Property (22) provides an exceptionally simple way of differentiating functions in the frequency domain—a valuable trick if you need to solve differential equations. Finally, equation (23) is known as the *Parseval's theorem* and it states that the *energy* of the signal is equal to the energy of the spectrum.

5 The spectrum

As you might have understood by now, the Fourier transform provides a *spectral representation* of a signal. It is also customary to speak of regarding the signal *in the time domain* or *in the frequency domain*, the latter corresponding to the Fourier transform of the signal. The two representations are equivalent as we are just observing the same signal in different bases, but the spectral representation is closer to the mechanics of sound generation and perception and thus often more convenient. For example, the *timbre* of a sound instrument can be best described by its spectrum.

The values $F(\nu)$ and $F(-\nu)$ of the spectrum show the phase and the amount of oscillation at frequency ν in the signal. If the signal is real-valued, then $F(\nu) = \overline{F(-\nu)}$, thus it often suffices to visualize only the positive part of the spectrum.

6 Windowed Fourier transform

One particular kind of Fourier transform you probably encounter most often is the *windowed Fourier transform (Gabor transform).* The idea is simple—when you are given a soundwave for a long musical composition, you are most often not really interested in the spectrum for this *whole* composition, but rather the spectrum of a small fragment of it near time t_0. So, naturally, you "cut out" the fragment of interest and perform the transform. This "cutting out" is done by multiplying the signal by a *windowing function* $\psi(t)$, which is often just a gaussian. The equation for the transform then looks like:

$$\mathcal{F}(t_0, \nu) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \nu t} \psi(t - t_0) \mathrm{d}t.$$

For real-life examples of Gabor transform just look at sheet music or note the jumping bars in WinAmp.

7 Discrete Fourier transform

Once sound gets into computer, it becomes digitalized. I.e. it's not a continuous signal $f(t)$ any more, but rather an N-dimensional vector. Generalization (or simplification?) of Fourier transform to this discrete case is known as the *discrete Fourier transform*, and now it *exactly* corresponds to a simple basis change we were talking about in equation (2). The corresponding equations are:

$$F_n = \sum_{k=0}^{N-1} f_k e^{-2\pi i n k/N} \qquad\qquad f_k = \frac{1}{N} \sum_{n=0}^{N-1} F_n e^{-2\pi i n k/N}$$

This form of Fourier transform satisfies the discrete analogs of all the important properties of the continuous version. An efficient algorithm exists (*Fast Fourier Transform, FFT*), that can compute the forward and inverse transforms in time $O(n \log n)$.

8 Why is all this useful?

Fourier theory has a multitude of uses in mathematics, statistics, data analysis and signal processing. Detection of periodicity in biological, financial, meteorological time series, solutions of differential equations, image processing and compression, sound synthesis, all kinds of pattern analysis issues—all that and much more are the areas of application of Fourier theory.