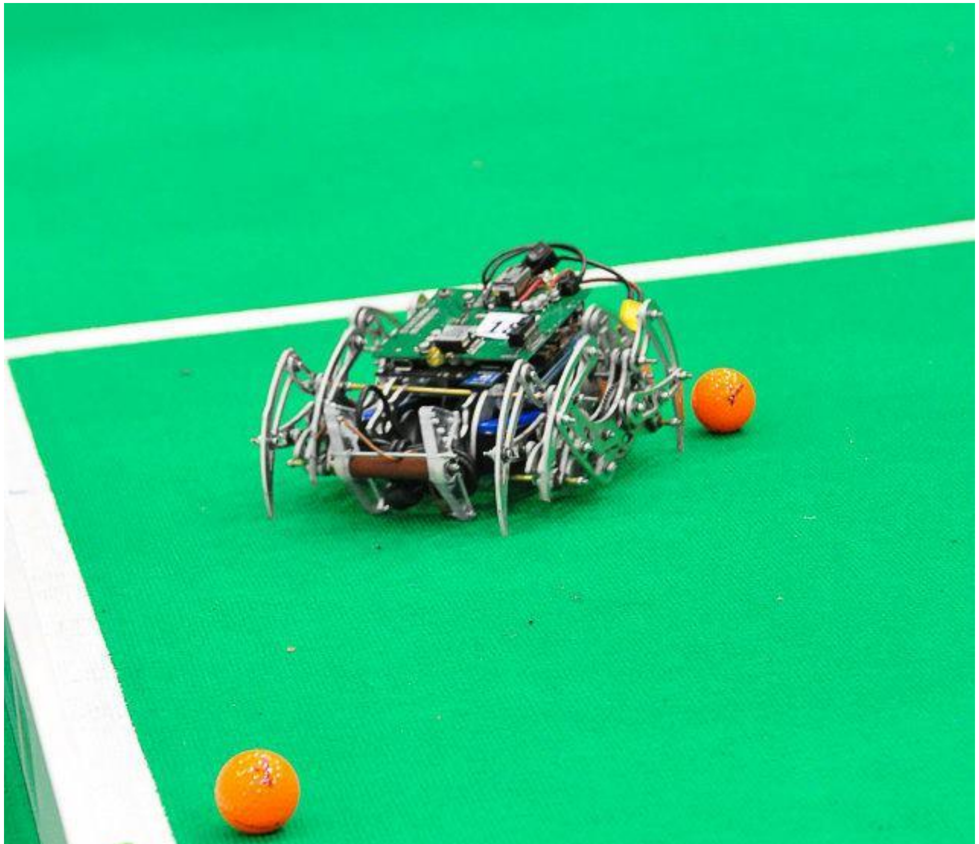
Probabilistic Localization

Konstantin Tretyakov (kt@ut.ee)



http://www.youtube.com/watch?v=0XbKZvXt5c4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TALLINNA
TEHNIKAÜL OL
GINEERIN
ELEKTER
MEHÜN
Ettevõtlusamet
ologia
i üliõpilasesindus
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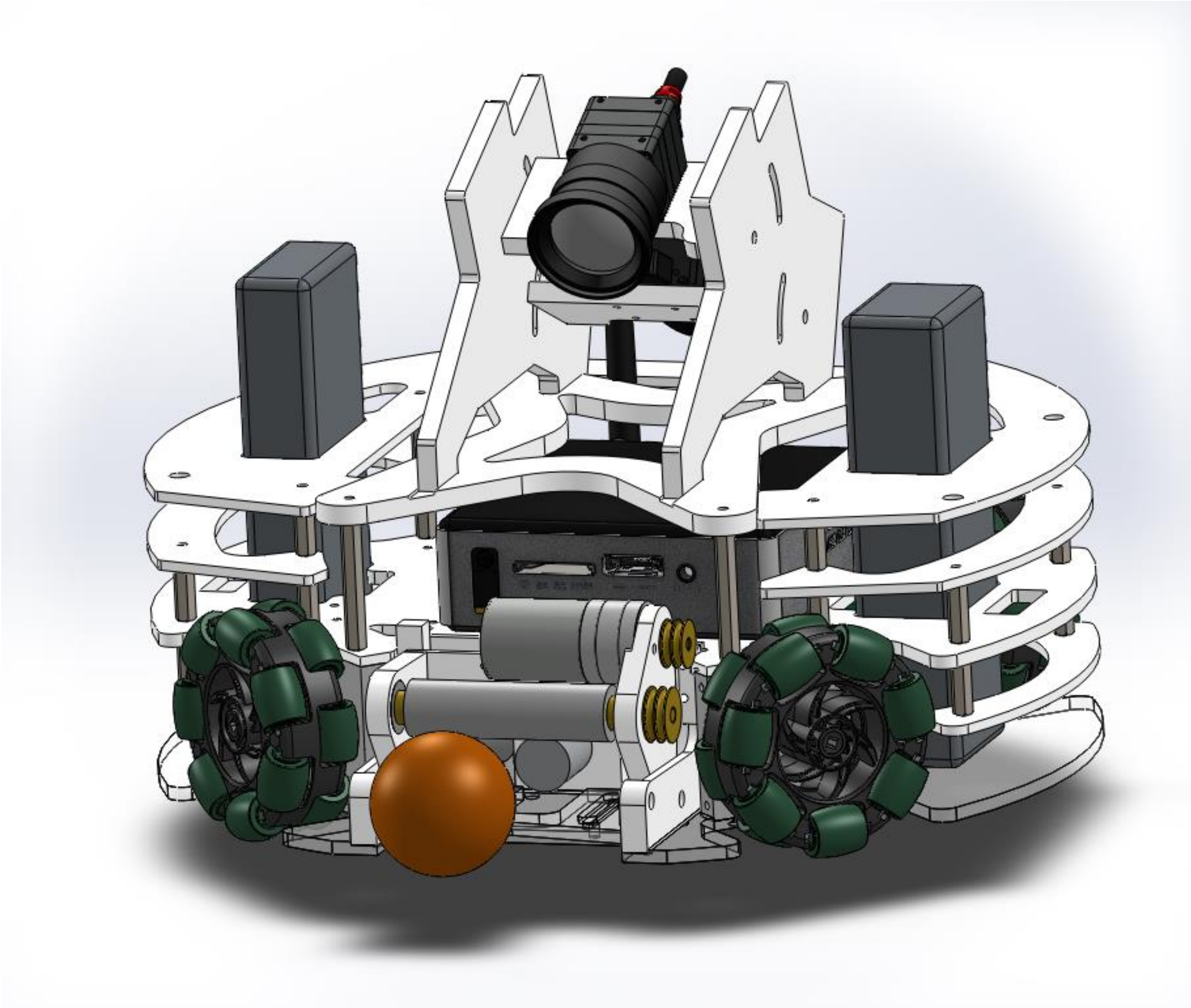
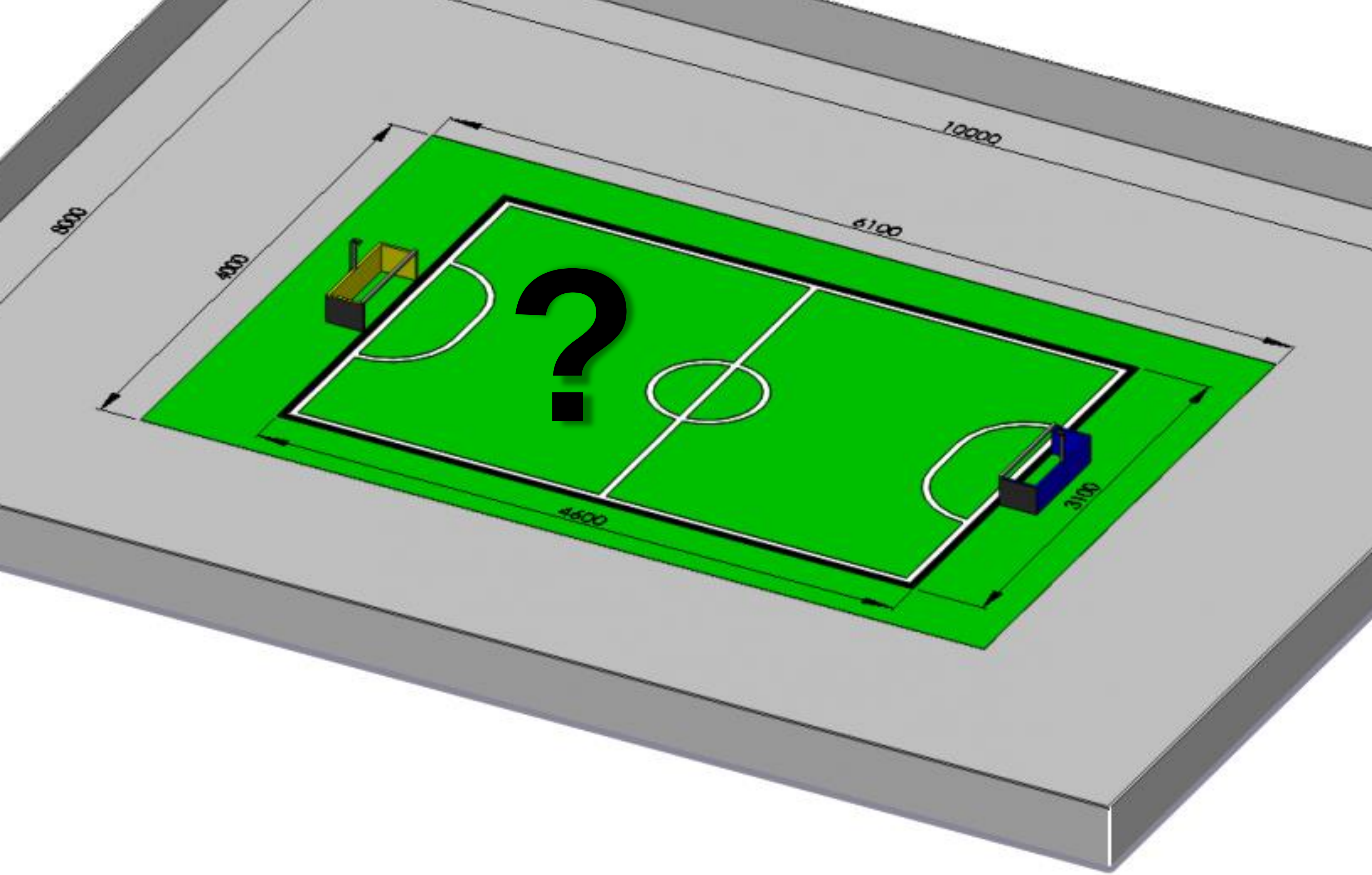




10000
6100
8000
4000
?
4600
3100

Three general problems

- **Localization**
- **Mapping**
- **“SLAM”**

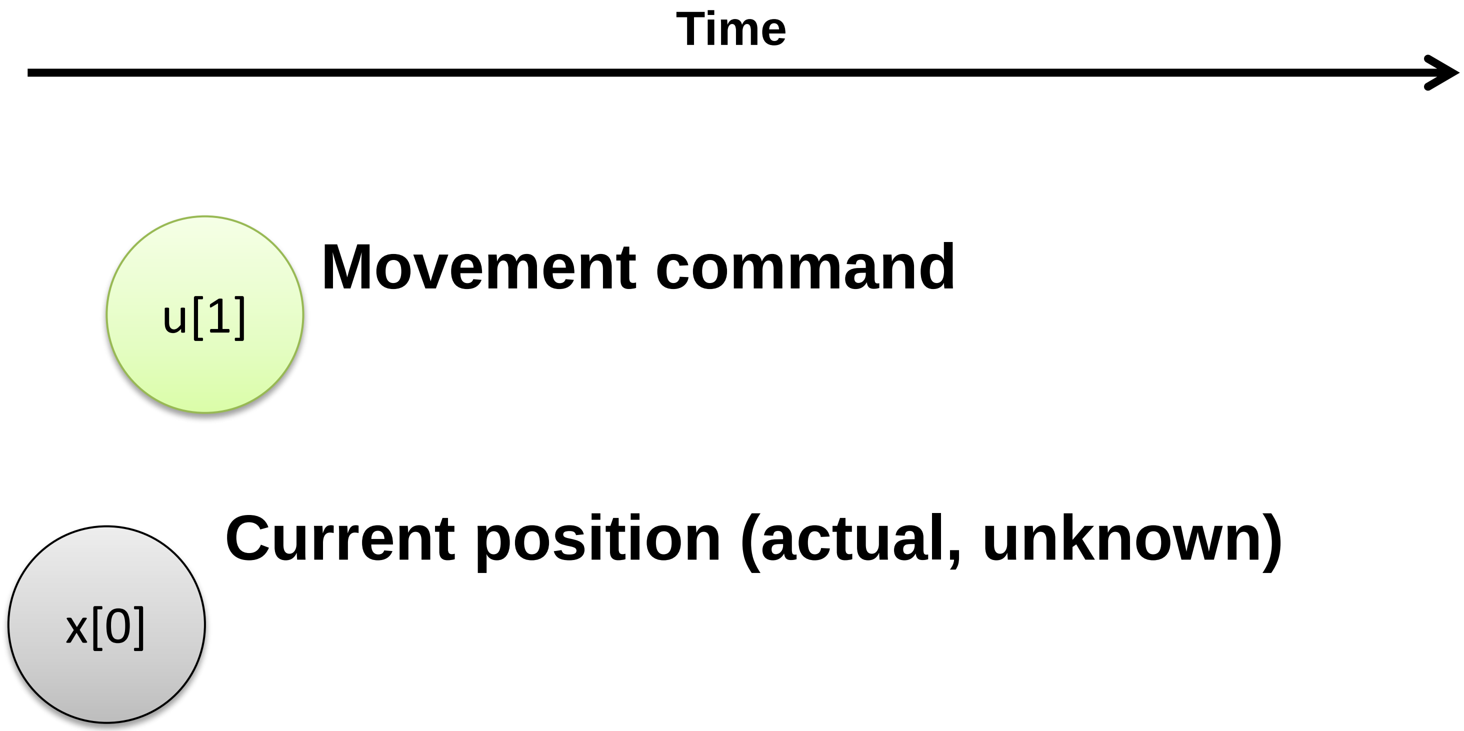
Our topic

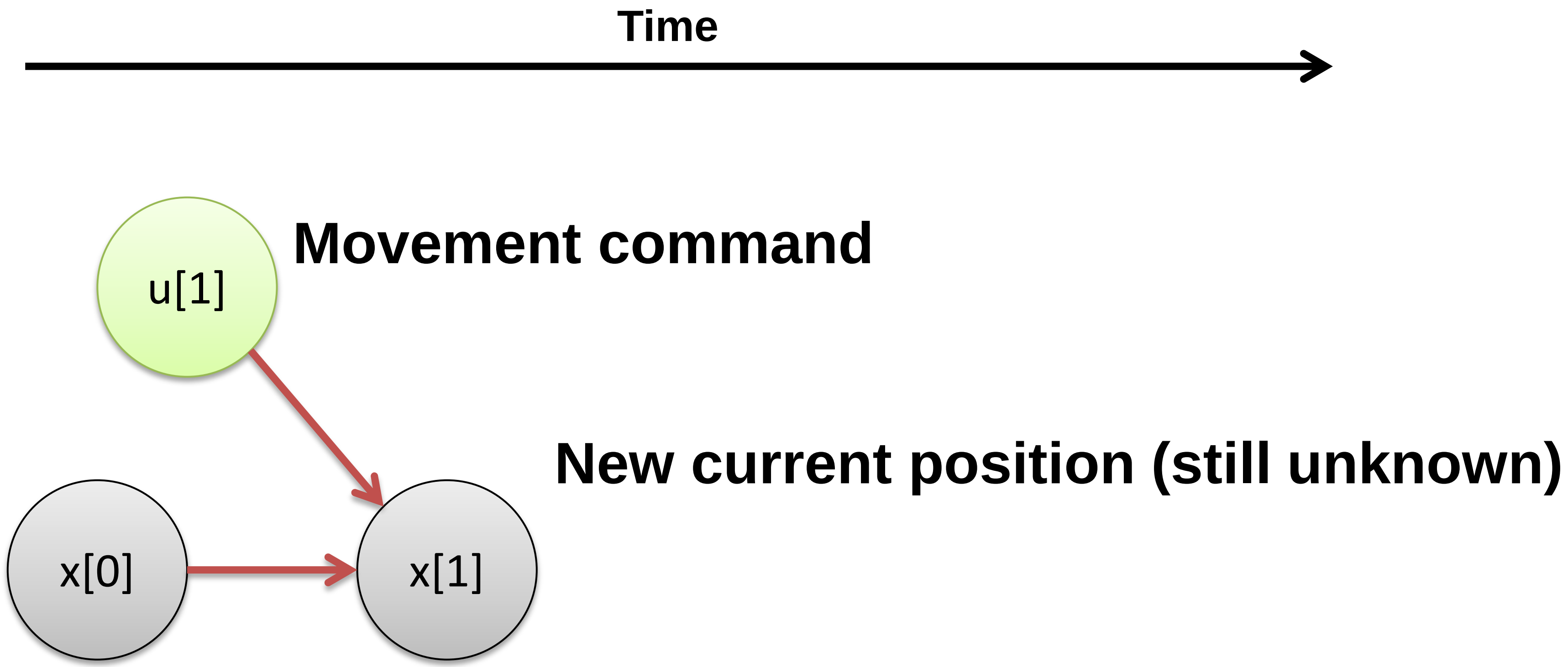
- **Localization**
- **Mapping**
- **“SLAM”**

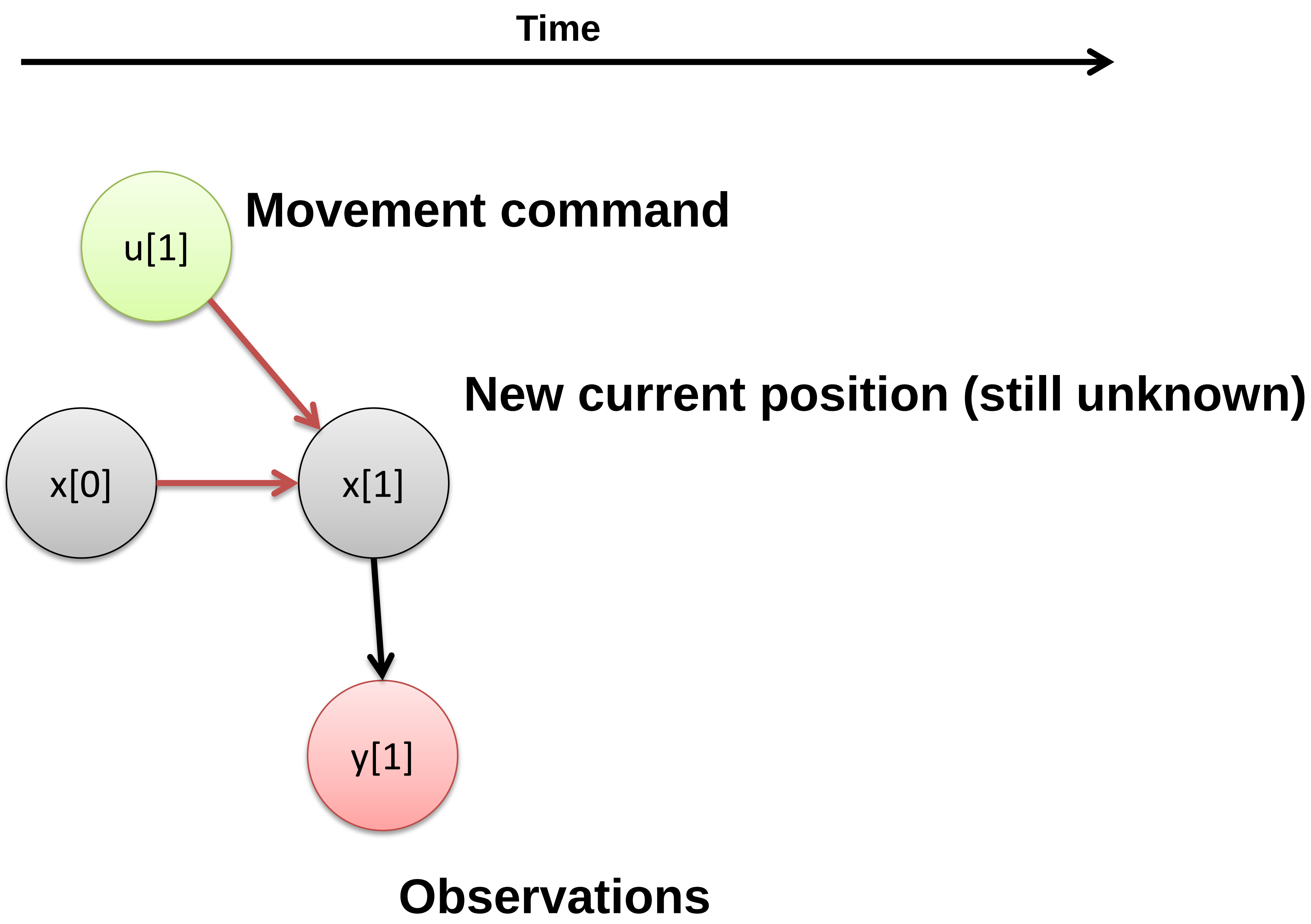
Time

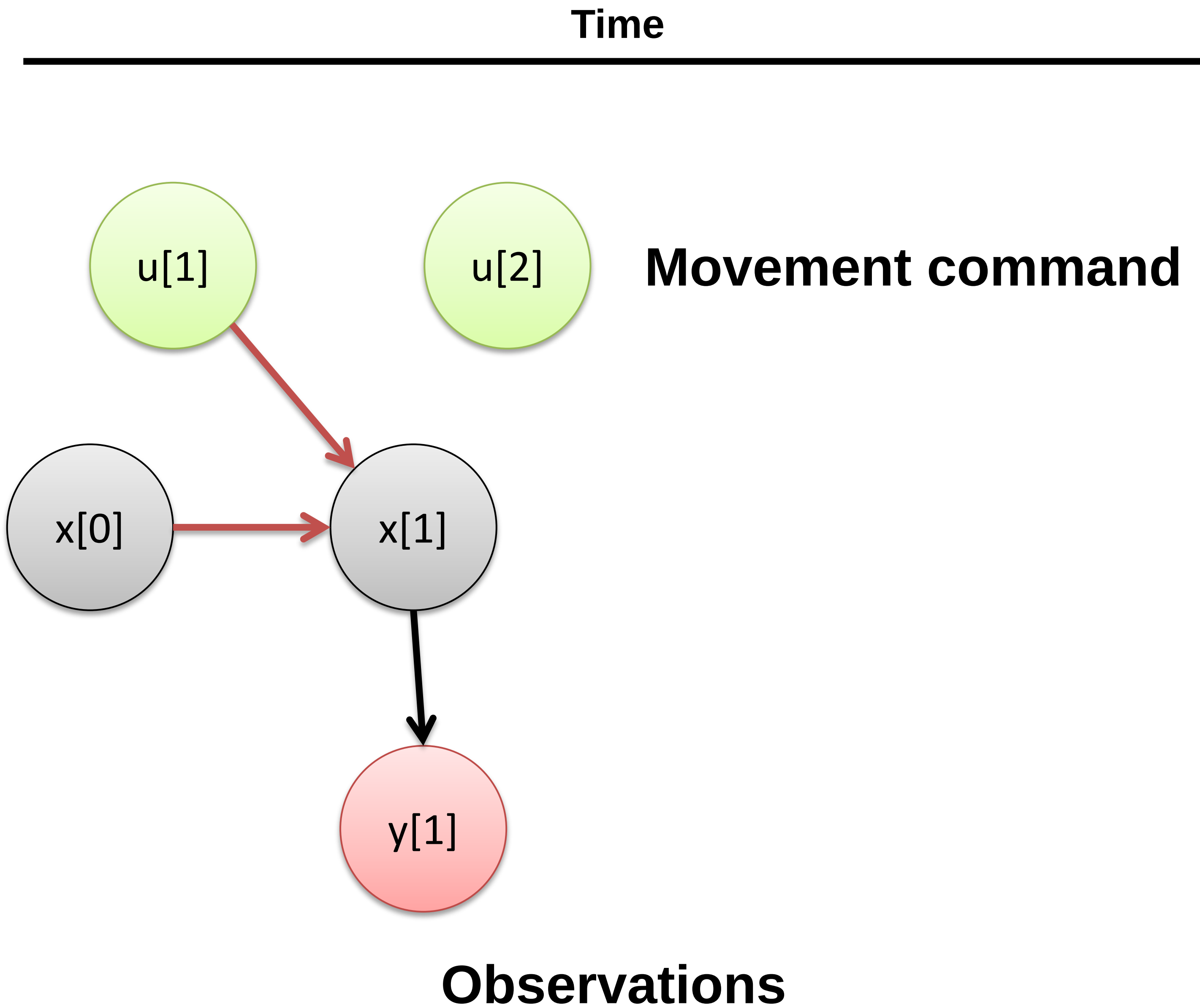
x[0]

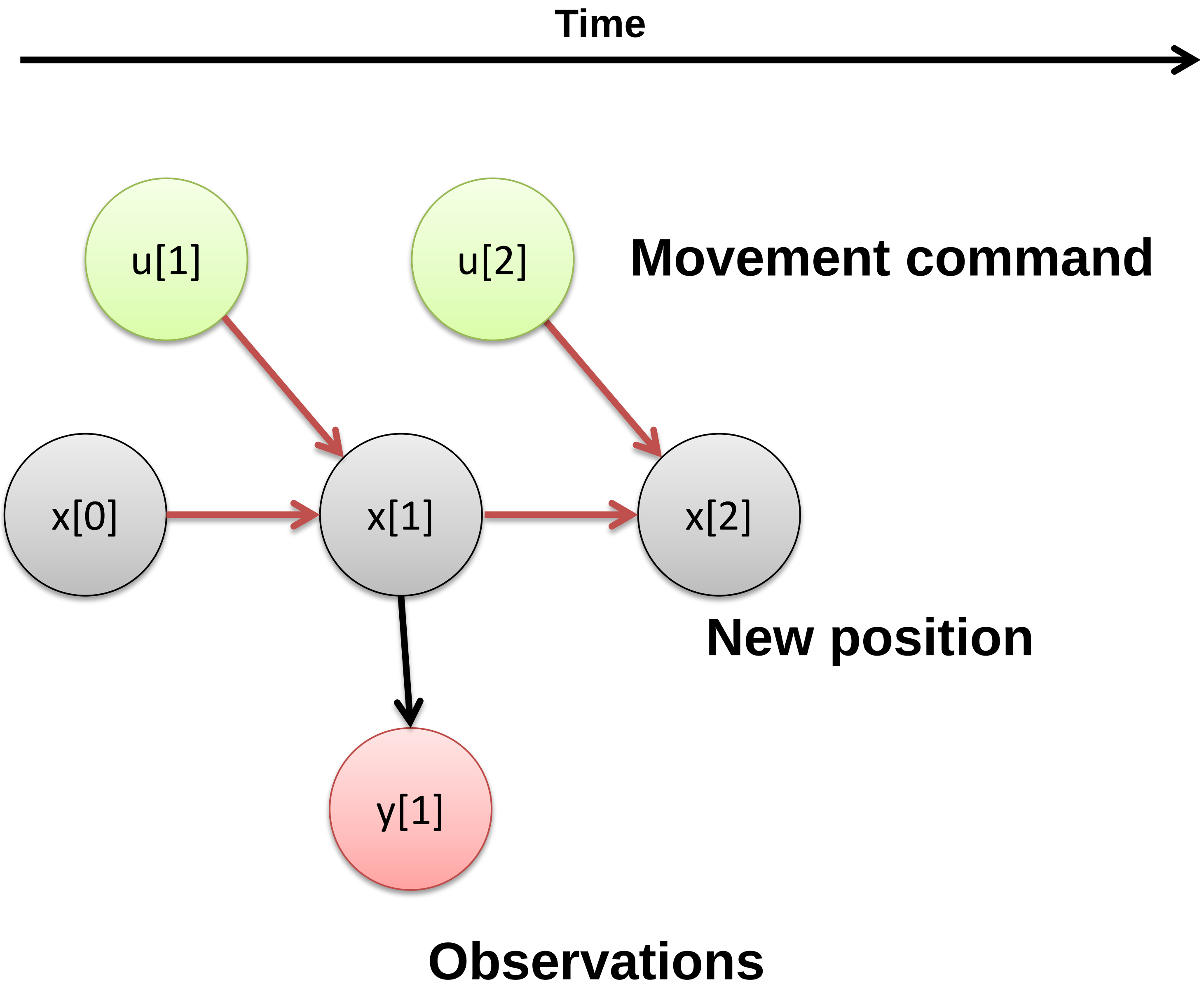
Current position (actual, unknown)

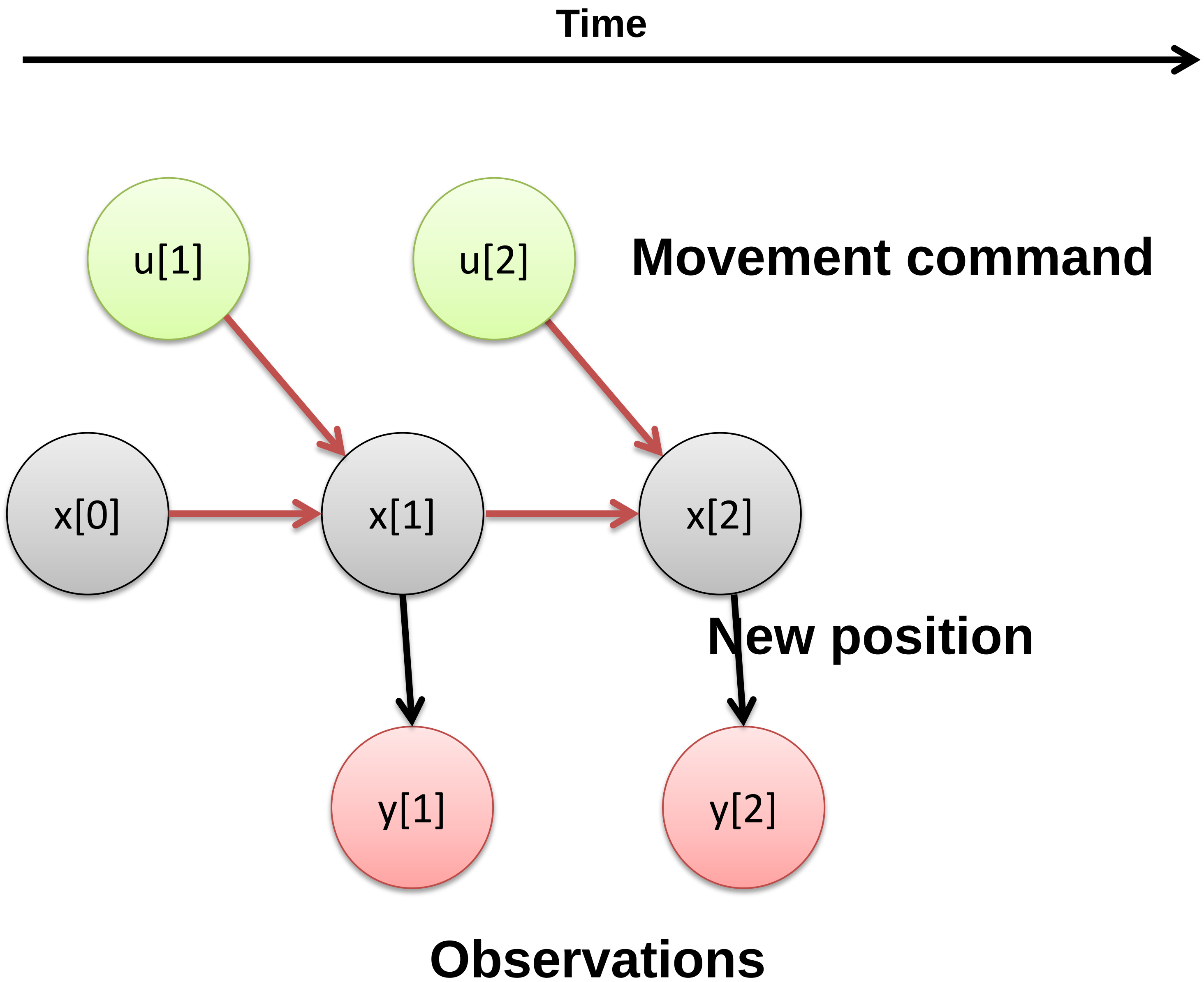

Time
u[1]
Movement command
x[0]
Current position (actual, unknown)

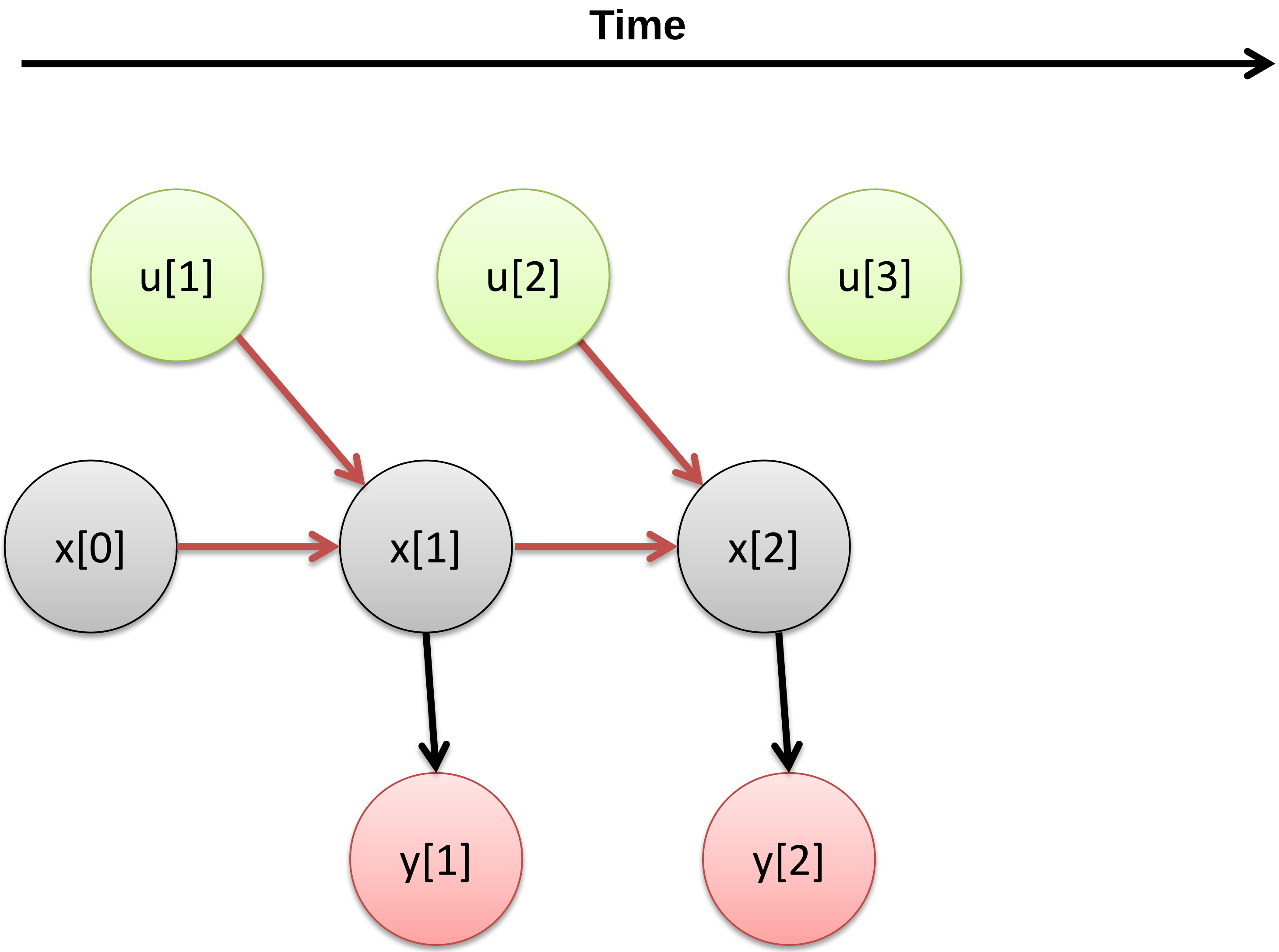

Time
u[1]
Movement command
New current position (still unknown)
x[0]
x[1]


Time
Movement command
u[1]
New current position (still unknown)
x[0]
x[1]
y[1]
Observations

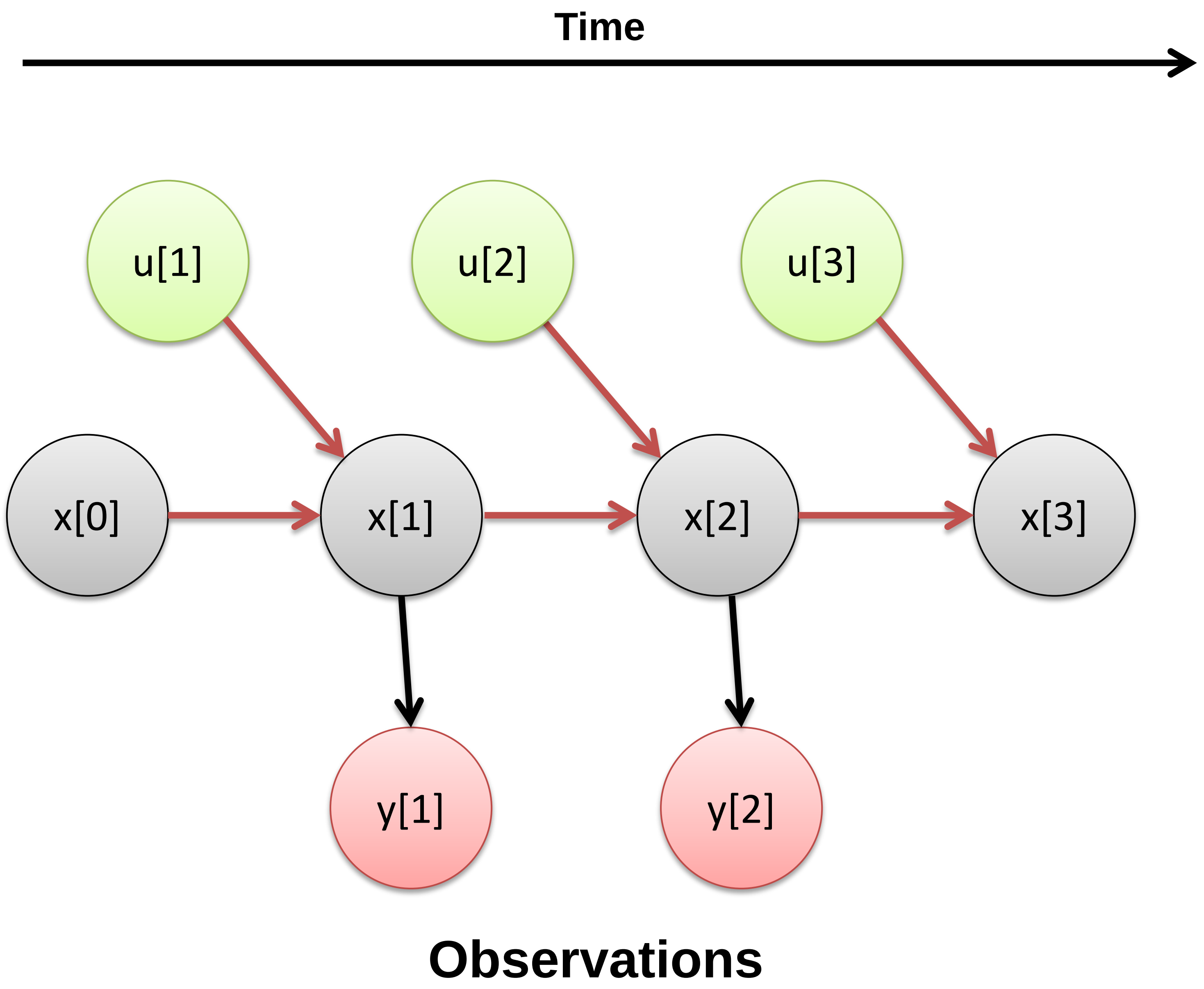

Time
u[1]
u[2]
Movement command
x[0]
x[1]
y[1]
Observations

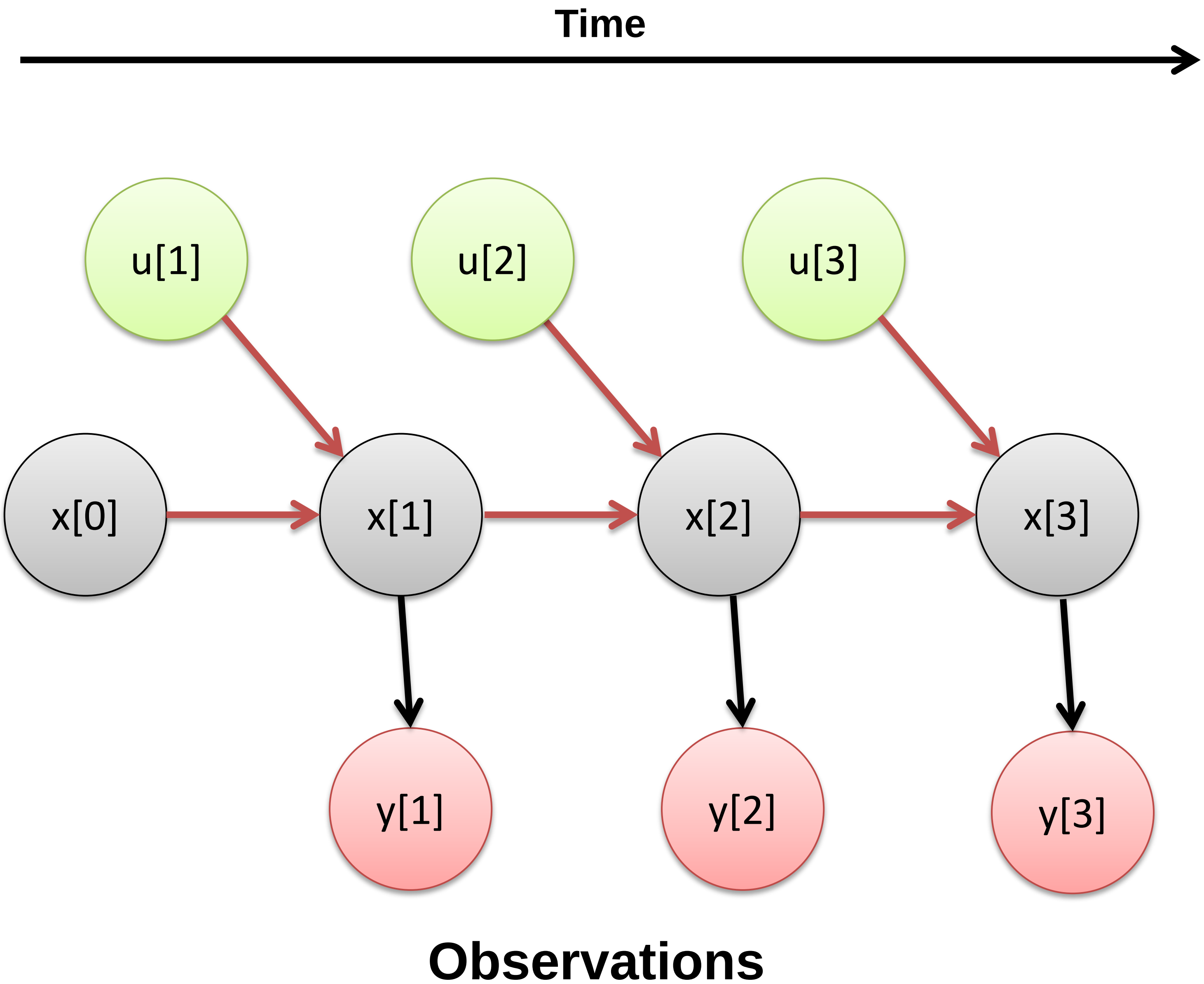

Time
u[1]
u[2]
Movement command
x[0]
x[1]
x[2]
New position
y[1]
Observations

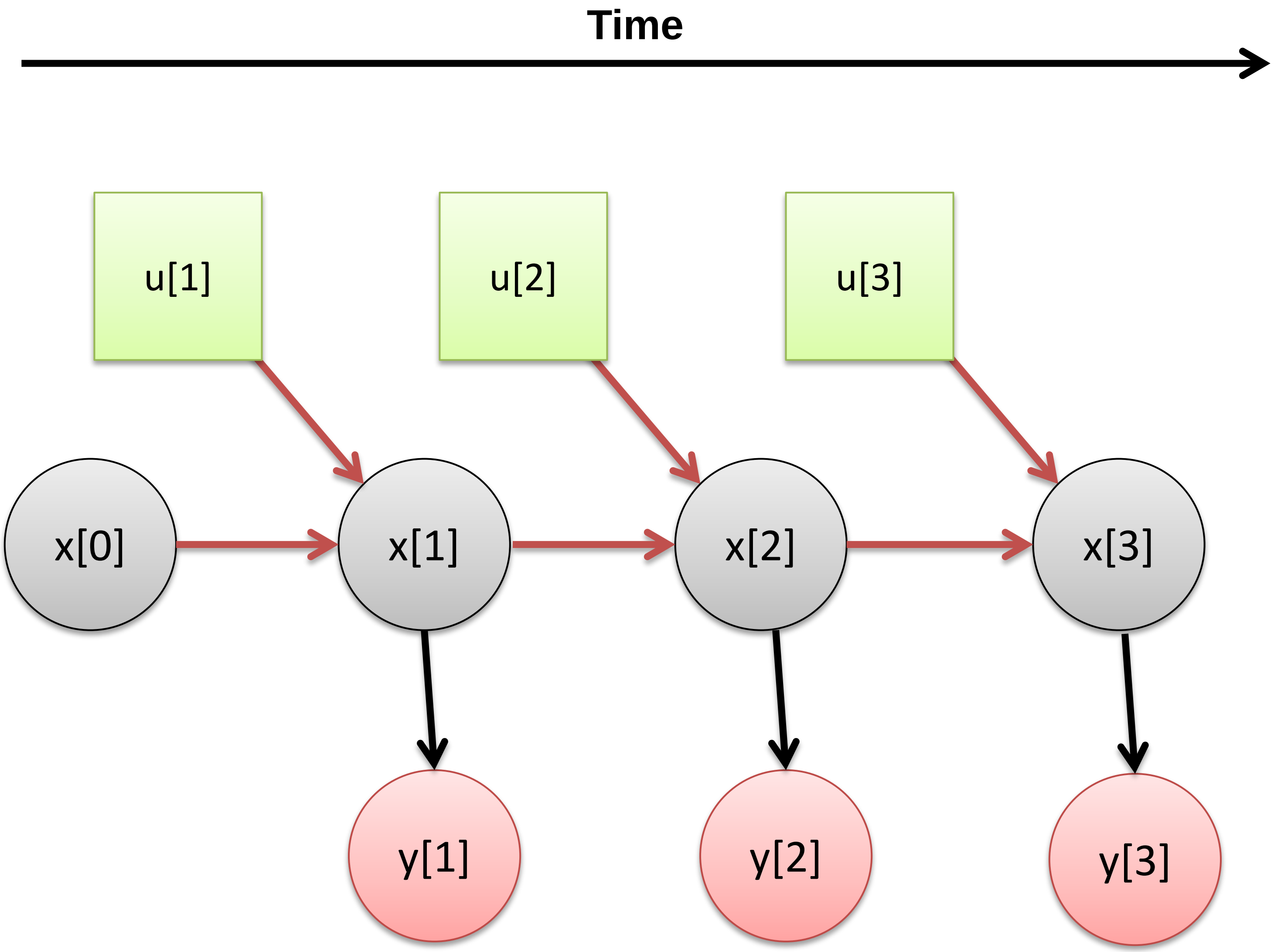

Time
u[1]
u[2]
Movement command
x[0]
x[1]
x[2]
New position
y[1]
y[2]
Observations



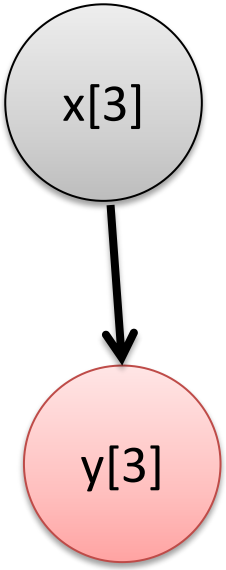
Observations

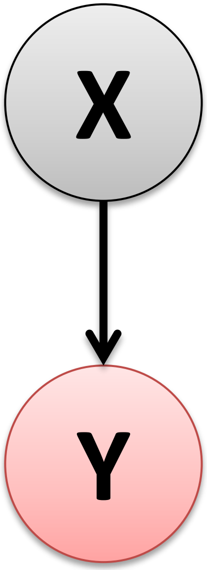

Time
u[1]
u[2]
u[3]
x[0]
x[1]
x[2]
x[3]
y[1]
y[2]
Observations

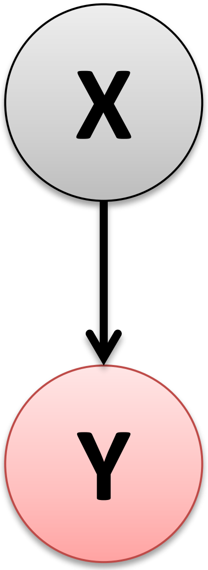

Time
u[1]
u[2]
u[3]
x[0]
x[1]
x[2]
x[3]
y[1]
y[2]
y[3]
Observations



(Temporal) Bayesian Network


x[3]
y[3]

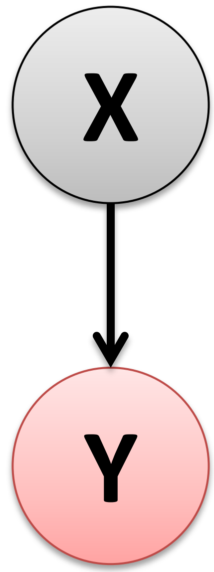

X
Y



$$\Pr[X]$$

$$\Pr[Y|X]$$

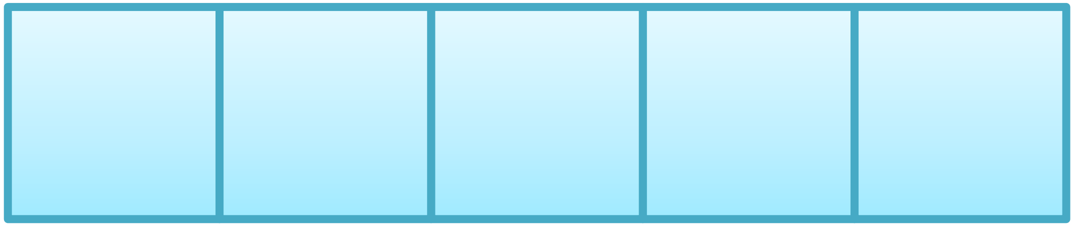
$$\Pr[X, Y] = \Pr[X] \cdot \Pr[Y|X]$$



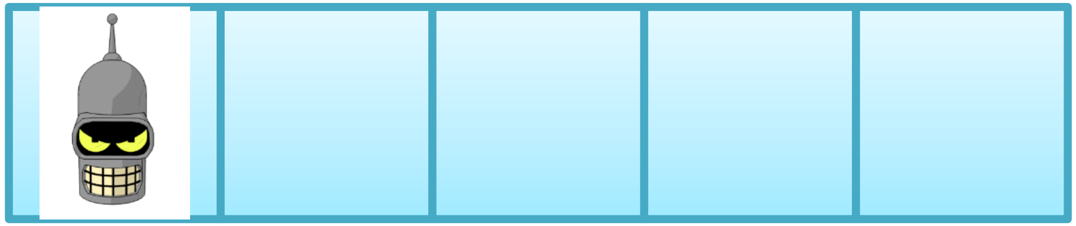
$$\Pr[X]$$

$$\Pr[Y|X]$$

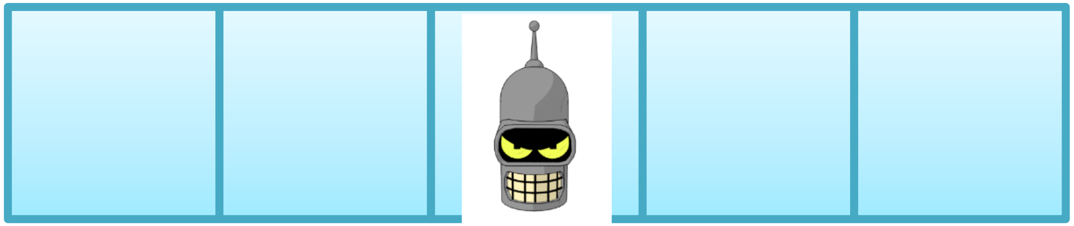
Probabilistic model



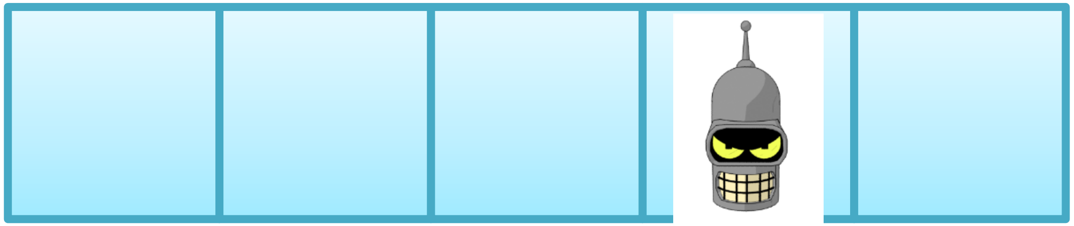
$$X \in \{1,2,3,4,5\}$$



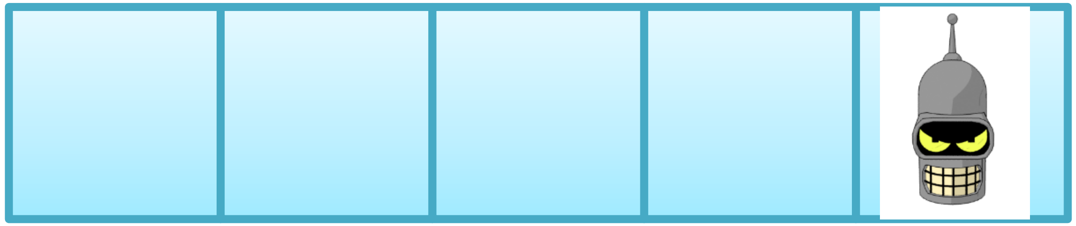
$$X \in \{1,2,3,4,5\}$$



$$X \in \{1,2,3,4,5\}$$



$$X \in \{1,2,3,4,5\}$$



$$X \in \{1,2,3,4,5\}$$

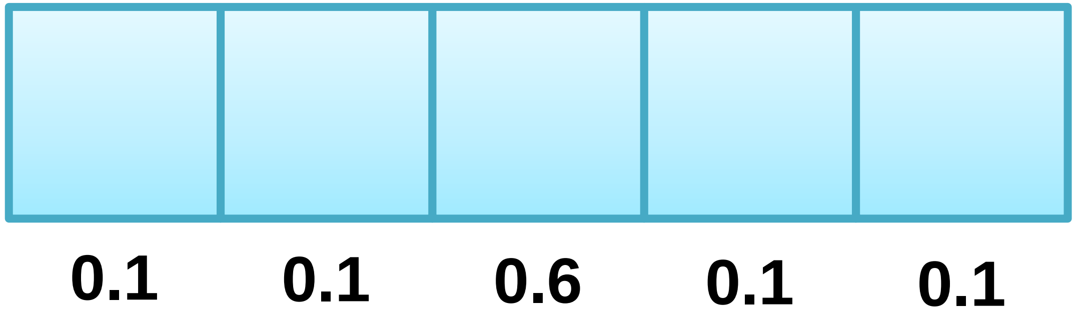
$$\Pr[X = \mathbf{3}] = \mathbf{0.6}$$

$$\Pr[X = 1] = 0.1$$

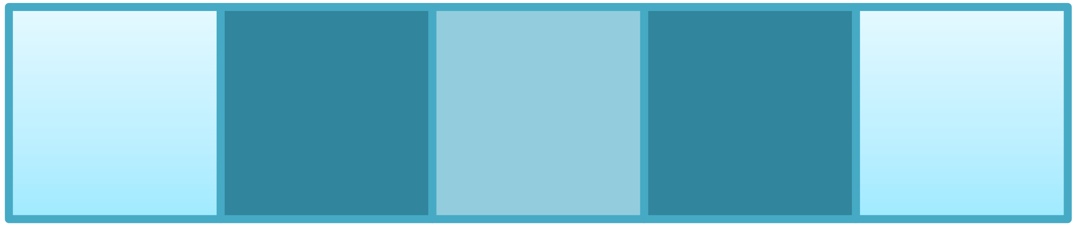
$$\Pr[X = 2] = 0.1$$

$$\Pr[X = 4] = 0.1$$

$$\Pr[X = 5] = 0.1$$



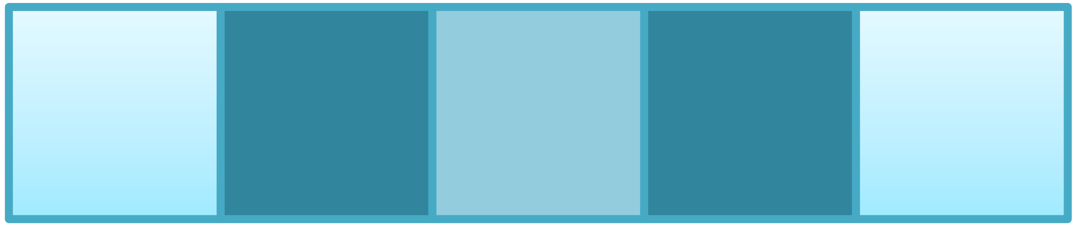
$$X \in \{1,2,3,4,5\}$$



0.1 0.1 0.6 0.1 0.1

$$X \in \{1,2,3,4,5\}$$

$$Y \in \{\text{dark}, \text{light}\}$$



0.1 0.1 0.6 0.1 0.1

$$X \in \{1,2,3,4,5\}$$

$$Y \in \{\text{dark}, \text{light}\}$$

$$\Pr[Y = \text{dark} | X = 1] = 0.2$$

$$\Pr[Y = \text{dark} | X = 2] = 0.9$$

$$\Pr[Y = \text{dark} | X = 3] = 0.5$$

20% dark	90% dark	50% dark	90% dark	20% dark
0.1	**0.1**	**0.6**	**0.1**	**0.1**

$$X \in \{1,2,3,4,5\}$$

$$Y \in \{\text{dark}, \text{light}\}$$

20% dark	90% dark	50% dark	90% dark	20% dark
0.1	**0.1**	**0.6**	**0.1**	**0.1**

$$X \in \{1,2,3,4,5\}$$

$$Y \in \{\text{dark}, \text{light}\}$$

$$\Pr[\text{dark} \mid 2] = ?$$

20% dark	90% dark	50% dark	90% dark	20% dark
0.1	0.1	0.6	0.1	0.1

$$X \in \{1,2,3,4,5\}$$

$$Y \in \{\text{dark}, \text{light}\}$$

$$\Pr[\text{light} \mid 5] = ?$$

20% dark	90% dark	50% dark	90% dark	20% dark
0.1	**0.1**	**0.6**	**0.1**	**0.1**

$$X \in \{1,2,3,4,5\}$$

$$Y \in \{\text{dark}, \text{light}\}$$

$$\Pr[\text{dark}, 1] = ?$$

20% dark	90% dark	50% dark	90% dark	20% dark
0.1	**0.1**	**0.6**	**0.1**	**0.1**

$$X \in \{1,2,3,4,5\}$$

$$Y \in \{\text{dark}, \text{light}\}$$

$$\Pr[4, \text{light}] = ?$$

20% dark	90% dark	50% dark	90% dark	20% dark
0.1	**0.1**	**0.6**	**0.1**	**0.1**

$$X \in \{1,2,3,4,5\}$$

$$Y \in \{\text{dark}, \text{light}\}$$

$$\Pr[4] = ?$$

20% dark	90% dark	50% dark	90% dark	20% dark
0.1	**0.1**	**0.6**	**0.1**	**0.1**

$$X \in \{1,2,3,4,5\}$$

$$Y \in \{\text{dark}, \text{light}\}$$

$$\Pr[\text{light}] = ?$$

20% dark	90% dark	50% dark	90% dark	20% dark
0.1	**0.1**	**0.6**	**0.1**	**0.1**

$$X \in \{1,2,3,4,5\}$$

$$Y \in \{\text{dark}, \text{light}\}$$

$$\Pr[2 \mid \text{light}] = ?$$

20% dark	90% dark	50% dark	90% dark	20% dark
0.1	**0.1**	**0.6**	**0.1**	**0.1**

$$X \in \{1,2,3,4,5\}$$

$$Y \in \{\text{dark}, \text{light}\}$$

$$\Pr[2 \mid \text{dark}] = ?$$

$$\Pr[X, Y] = \Pr[Y]\Pr[X|Y]$$

$$\Pr[X, Y] = \Pr[Y] \Pr[X|Y]$$

$$\Pr[X|Y] = ?$$

$$\Pr[X,Y] = \Pr[Y]\Pr[X|Y]$$

$$\Pr[X|Y] = \frac{\Pr[\mathrm{X},\mathrm{Y}]}{\Pr[\mathrm{Y}]}$$

$$\Pr[X, Y] = \Pr[Y] \Pr[X|Y]$$

$$\Pr[X|Y] = \frac{\Pr[\mathrm{Y}|\mathrm{X}] \Pr[\mathrm{X}]}{\Pr[\mathrm{Y}]}$$

The Bayes Rule

$$\Pr[X|Y] = \frac{\Pr[\mathrm{Y}|\mathrm{X}]\Pr[\mathrm{X}]}{\Pr[\mathrm{Y}]}$$

The Bayes Rule

$$\Pr[X|Y] = \frac{\Pr[Y|X]\Pr[X]}{\Pr[Y]}$$

"Prior"

The Bayes Rule

$$\Pr[X|Y] = \frac{\Pr[Y|X]\Pr[X]}{\Pr[Y]}$$

"Prior"

"Posterior"

The Bayes Rule

$$\Pr[X|Y] = \frac{\Pr[\mathrm{Y}|\mathrm{X}]\Pr[\mathrm{X}]}{\Pr[\mathrm{Y}]}$$

“Posterior” — $\Pr[X|Y]$

“Observation likelihood” — $\Pr[\mathrm{Y}|\mathrm{X}]$

“Prior” — $\Pr[\mathrm{X}]$

20% dark	90% dark	50% dark	90% dark	20% dark
0.1	**0.1**	**0.6**	**0.1**	**0.1**

$$X \in \{1,2,3,4,5\}$$

$$Y \in \{\text{dark}, \text{light}\}$$

20% dark	90% dark	50% dark	90% dark	20% dark
0.1	**0.1**	**0.6**	**0.1**	**0.1**

$$X \in \{1,2,3,4,5\}$$

$$Y \in \{\text{dark}, \text{light}\}$$

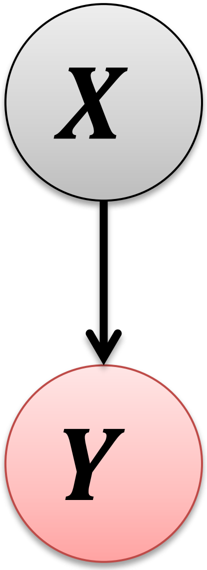
**I observed “dark”.
Where am I?**

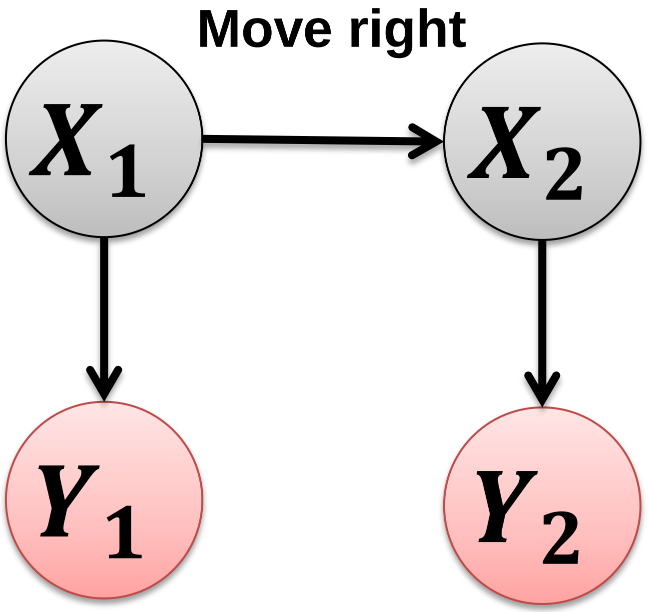
20% dark	90% dark	50% dark	90% dark	20% dark
0.1	**0.1**	**0.6**	**0.1**	**0.1**

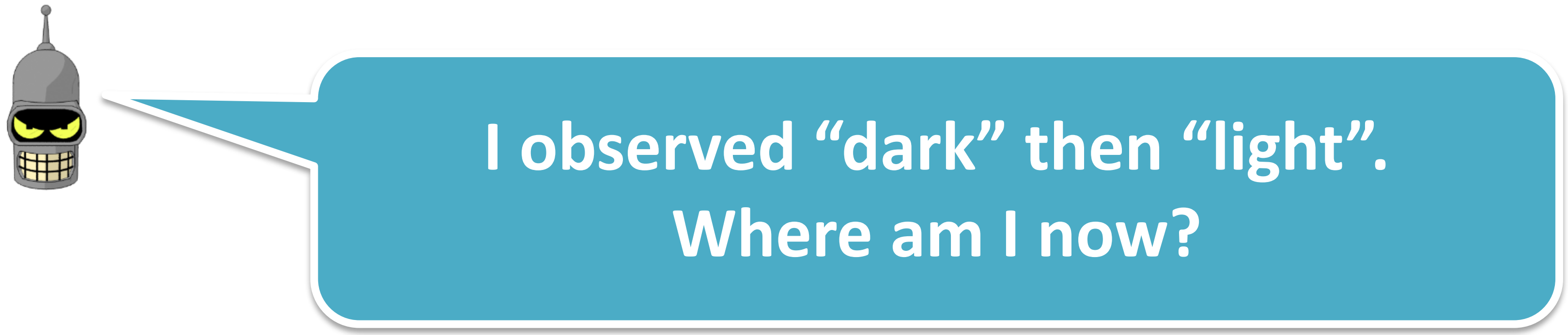
$$X \in \{1,2,3,4,5\}$$

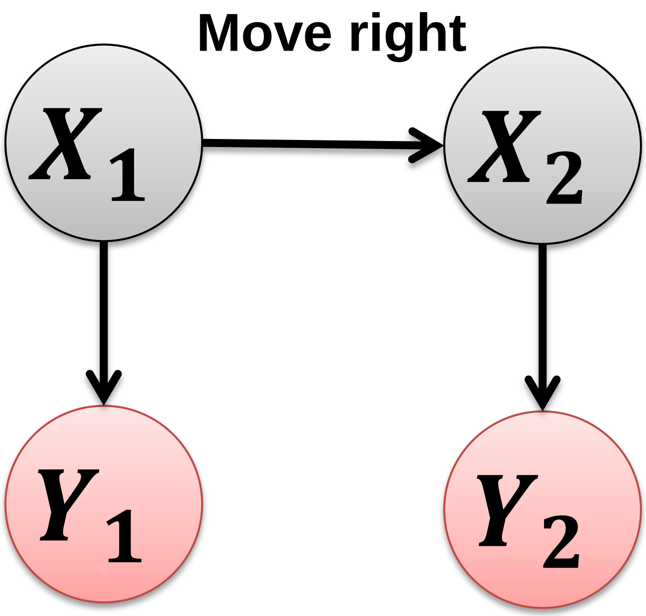
$$Y \in \{\text{dark}, \text{light}\}$$

I observed “light”.
Where am I?

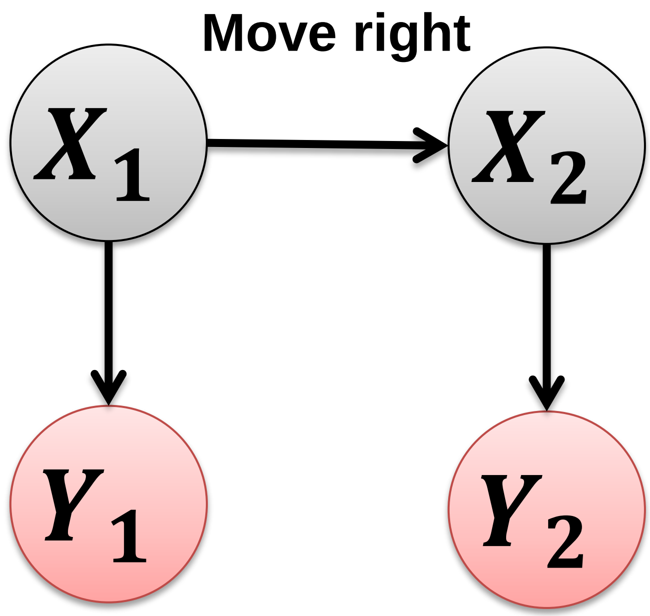

X
Y


Move right
X_1
X_2
Y_1
Y_2


I observed "dark" then "light".
Where am I now?

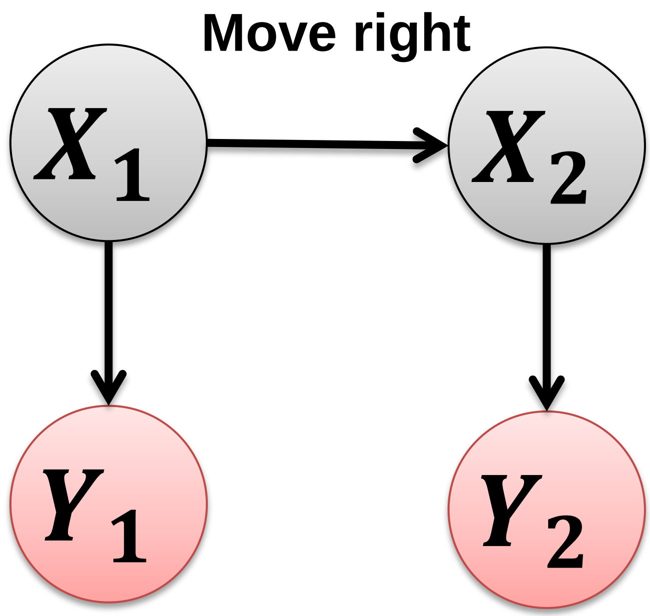

Move right
X_1
X_2
Y_1
Y_2

$$Pr[X_2 | Y_1 = \text{dark}, Y_2 = \text{light}]$$



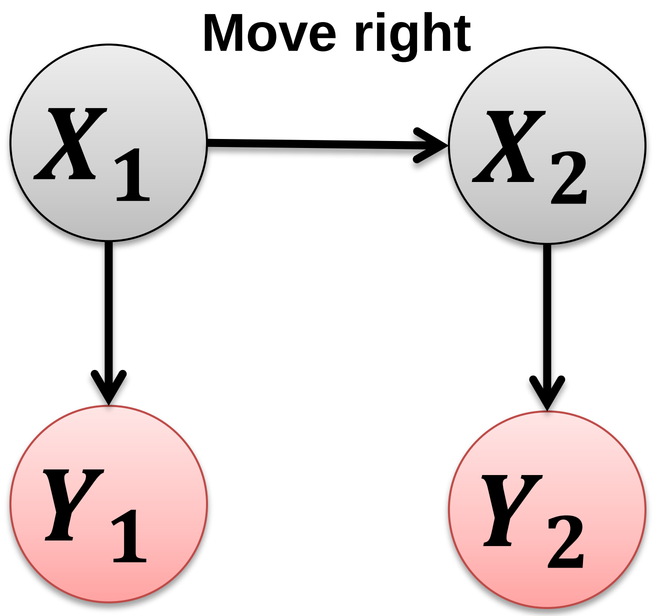
$$Pr[X_2 | Y_1 = \mathbf{dark}, Y_2 = \mathbf{light}]$$

$$\Pr[X_1]$$



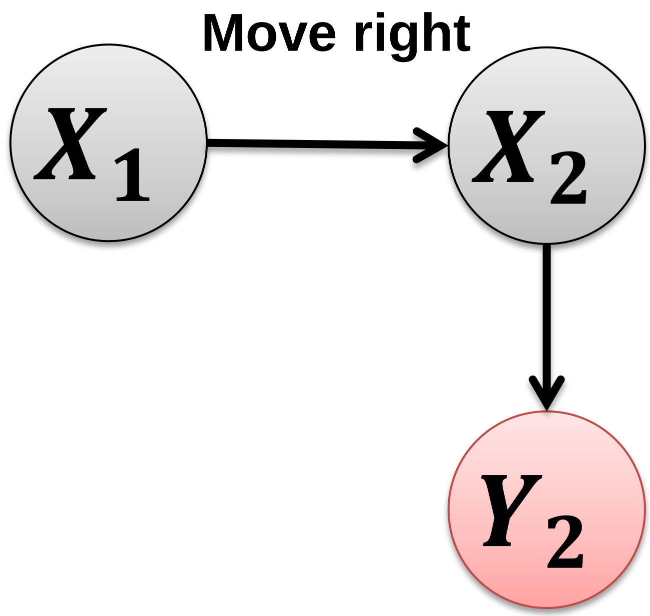
$$Pr[X_2|Y_1 = \text{dark}, Y_2 = \text{light}]$$

$$\Pr[X_1|Y_1]$$



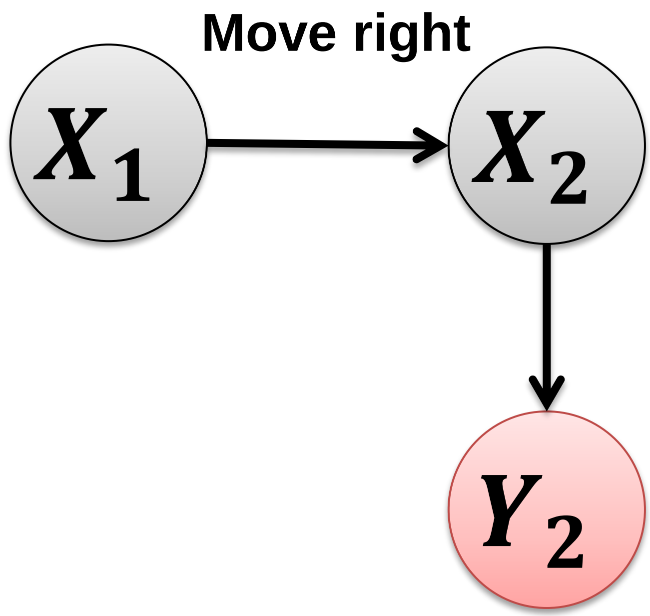
$$Pr[X_2 | Y_1 = \mathbf{dark}, Y_2 = \mathbf{light}]$$

$$\Pr[X_1 | Y_1]$$



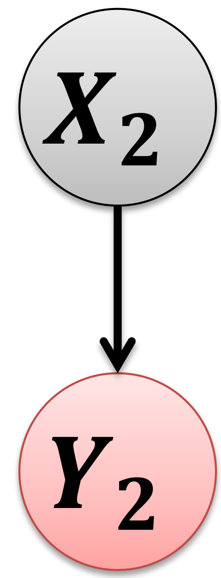
$$Pr[X_2|Y_1 = \text{dark}, Y_2 = \text{light}]$$

$$\Pr[X_1|Y_1]$$



$$\Pr[X_2|Y_1]$$

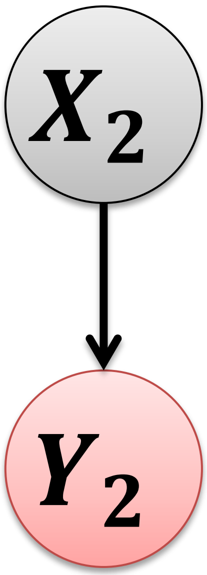
$$Pr[X_2 | Y_1 = \text{dark}, Y_2 = \text{light}]$$



$$\Pr[X_2 | Y_1]$$

$$\boldsymbol{Pr[X_2 | Y_1 = \text{dark}, Y_2 = \text{light}]}$$

$$\Pr[X_2 | Y_1, Y_2]$$



Bayesian Filtering

- Initialize prior distribution $\Pr[X]$
- Repeat forever:
  - **Observe $\boldsymbol{y}$**
    - $\Pr[X|\boldsymbol{y}] \propto \Pr[\boldsymbol{y}|X] \cdot \Pr[X]$, then normalize
  - **Move**
    - $\Pr[\mathrm{newX}] := \sum_{\mathrm{oldX}} \Pr[\mathrm{newX}|\mathrm{oldX}] \cdot \Pr[\mathrm{oldX}|\boldsymbol{y}]$

Discrete Bayesian Filtering

- Initialize prior distribution vector p
- Repeat forever:
  - **Observe y, then for all i**
    - $p[i] \coloneqq p[i] \cdot \mathrm{Likelihood}(y|i)$
    - $p[i] \coloneqq p[i]/\mathrm{sum}(p)$
  - **Move, i.e. for all i**
    - $p[i] \coloneqq \sum_j p[j] \cdot \mathrm{MoveProbability}(j \rightarrow i)$

Continuous Bayesian Filtering

- Initialize distribution function f
- Repeat forever:
  - **Observe y, then for all x**
    - $f(x) \coloneqq f(x) \cdot \mathrm{Likelihood}(y|x)$
    - $f(x) \coloneqq f(x) / \int f(x) dx$
  - **Move, i.e. for all x**
    - $f(x) \coloneqq \int_z f(z) \cdot \mathrm{MoveProbability}(z \to x) dz$

Continuous Bayesian Filtering

Continuous Bayesian Filtering

- Initialize distribution function f
- Repeat forever:
  - **Observe y, then for all x**
    - $f(x) := f(x) \cdot \mathrm{Likelihood}(y|x)$
    - $f(x) := f(x) / \int f(x) dx$
  - **Move, i.e. for all x**
    - $f(x) := \int_z f(z) \cdot \mathrm{MoveProbability}(z \to x) dz$

How to handle non-discrete distributions?

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- **Parameterized function families**
- **Population-based representation**

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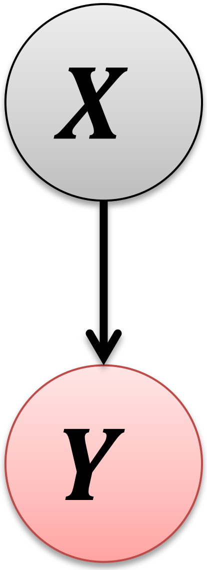
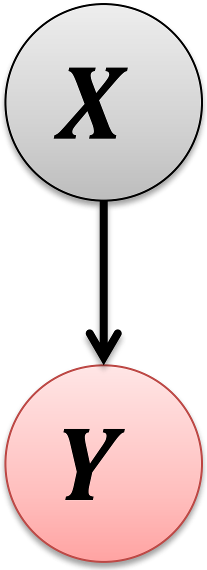
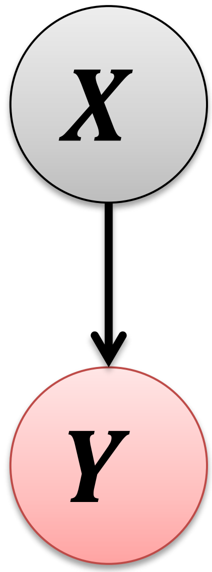
- **Parameterized function families**
- **Population-based representation**


X
Y

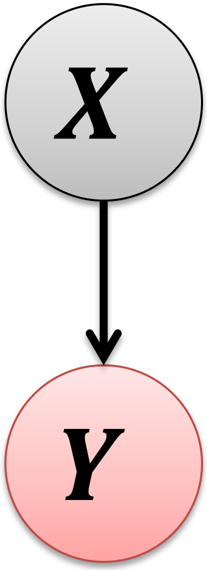


$$X \in \mathbb{R}$$

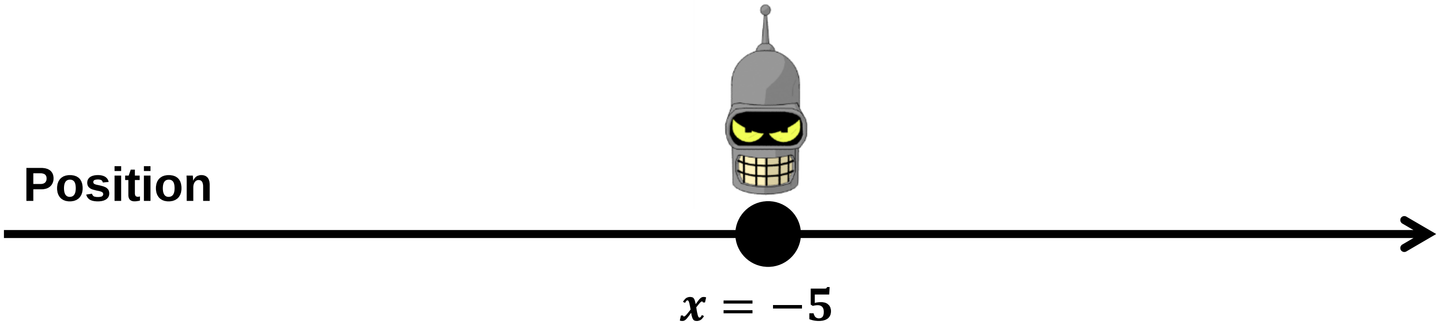
Position

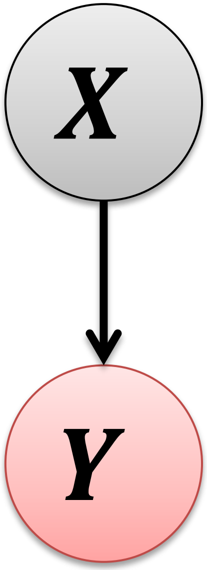


$$X \in \mathbb{R}$$



$$X \in \mathbb{R}$$



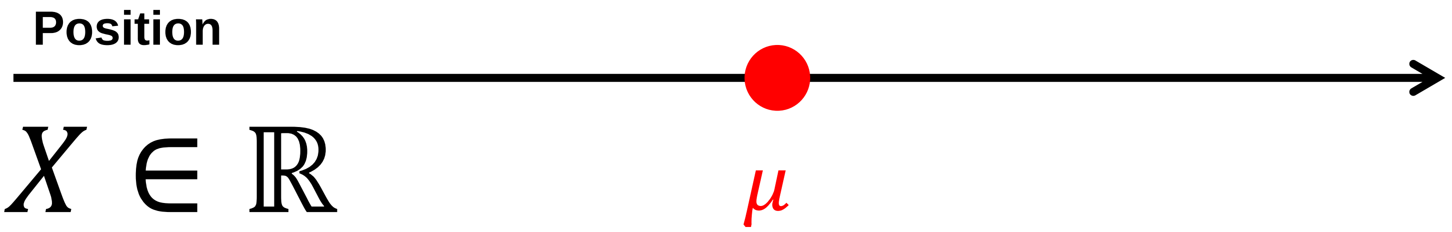


$$X \in \mathbb{R}$$

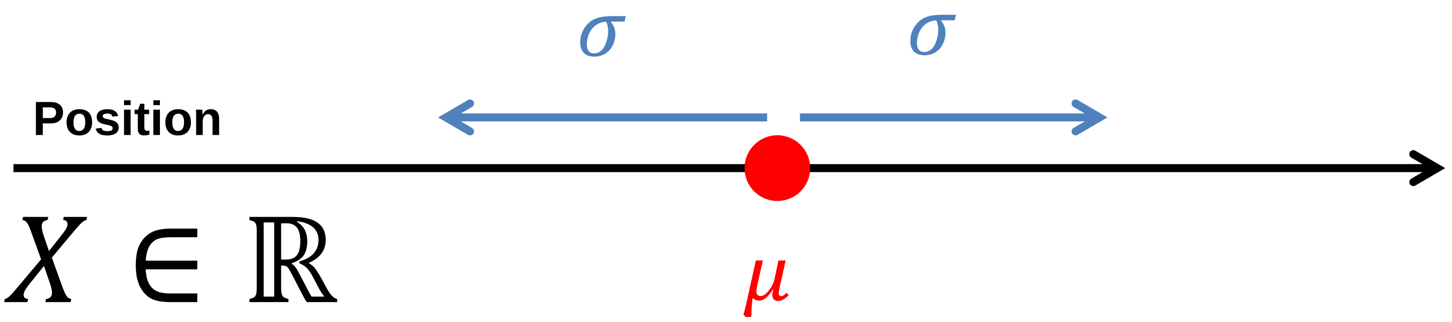
Position

$$X \in \mathbb{R}$$

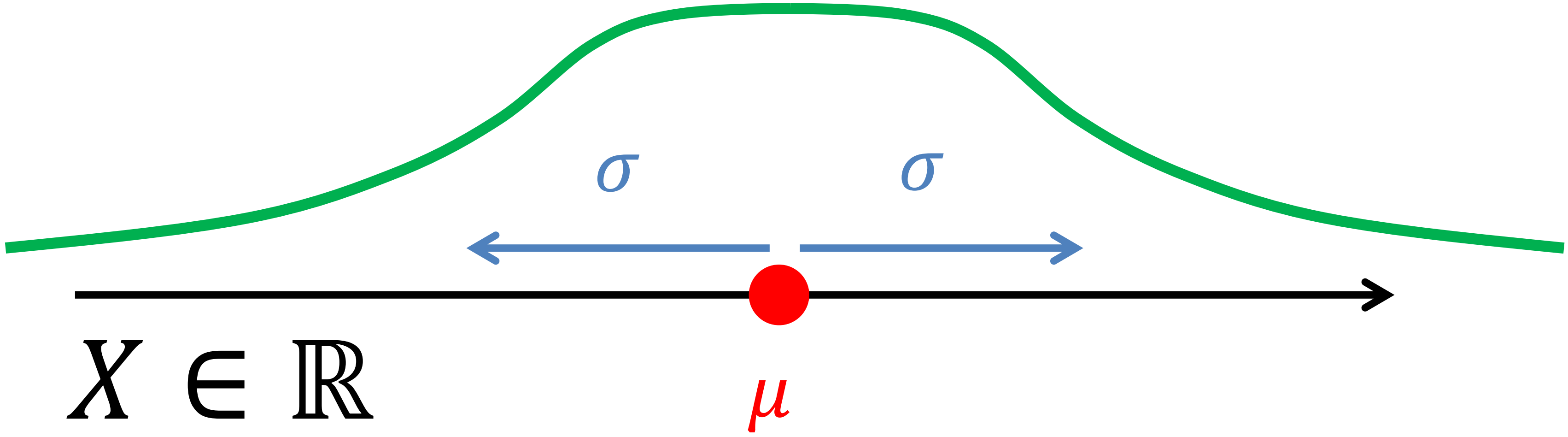
$$\Pr[X = x] = \alpha \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$



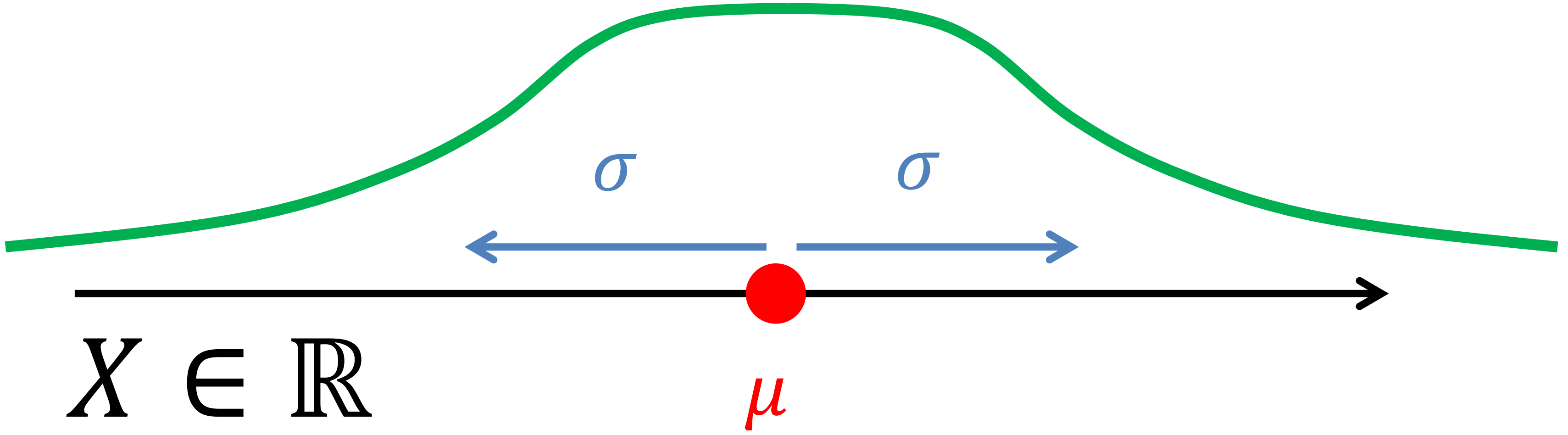
$$\Pr[X = x] = \alpha \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$



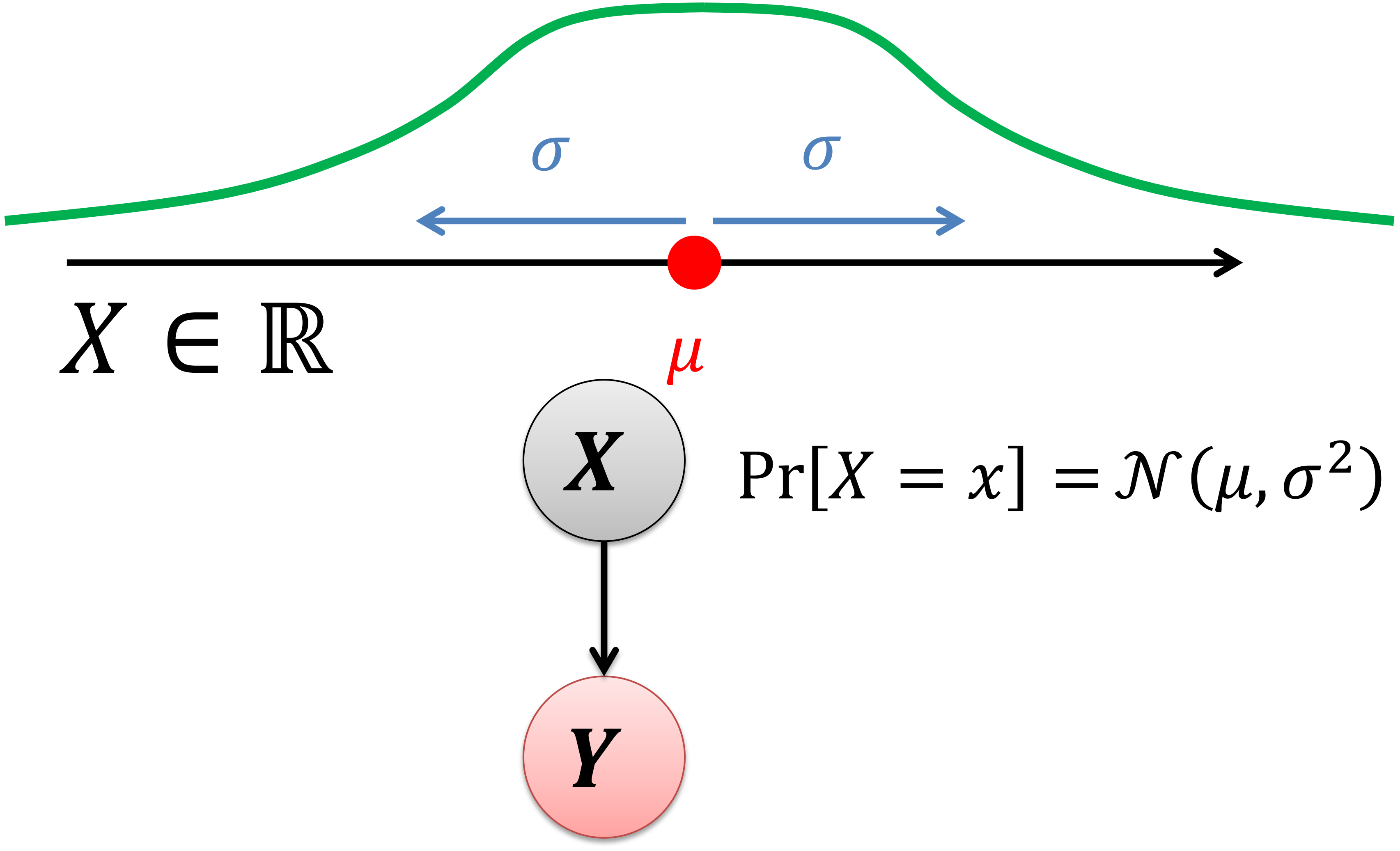
$$\Pr[X = x] = \alpha \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$



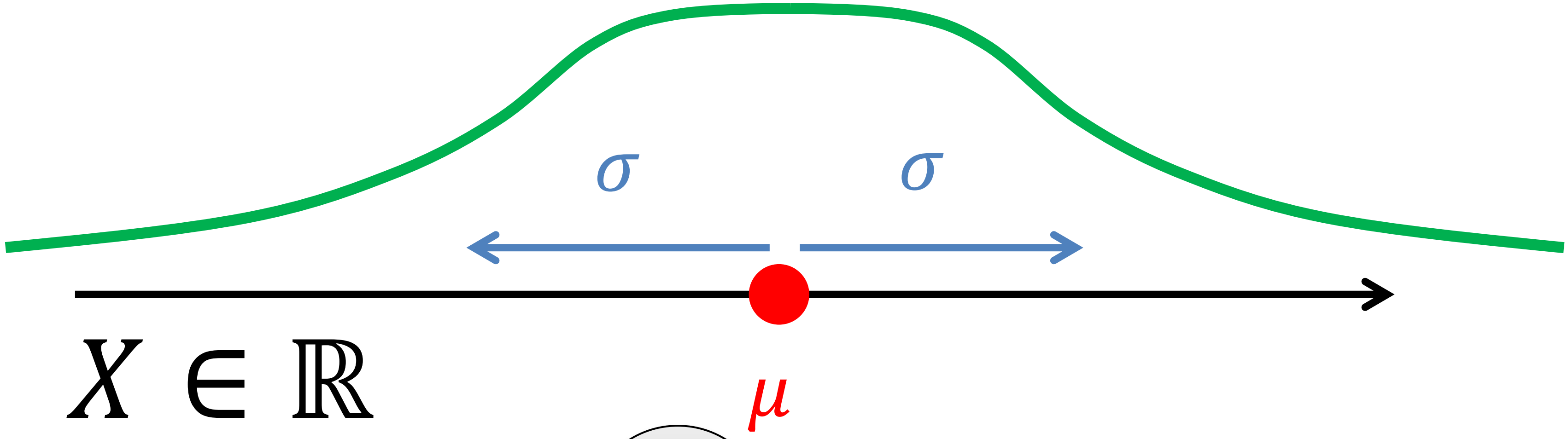
$$\Pr[X = x] = \alpha \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$



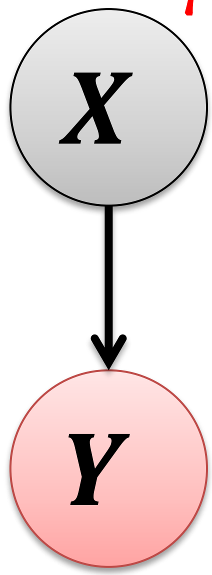
$$\Pr[X = x] = \mathcal{N}(\mu, \sigma^2)$$



$$\Pr[X = x] = \mathcal{N}(\mu, \sigma^2)$$

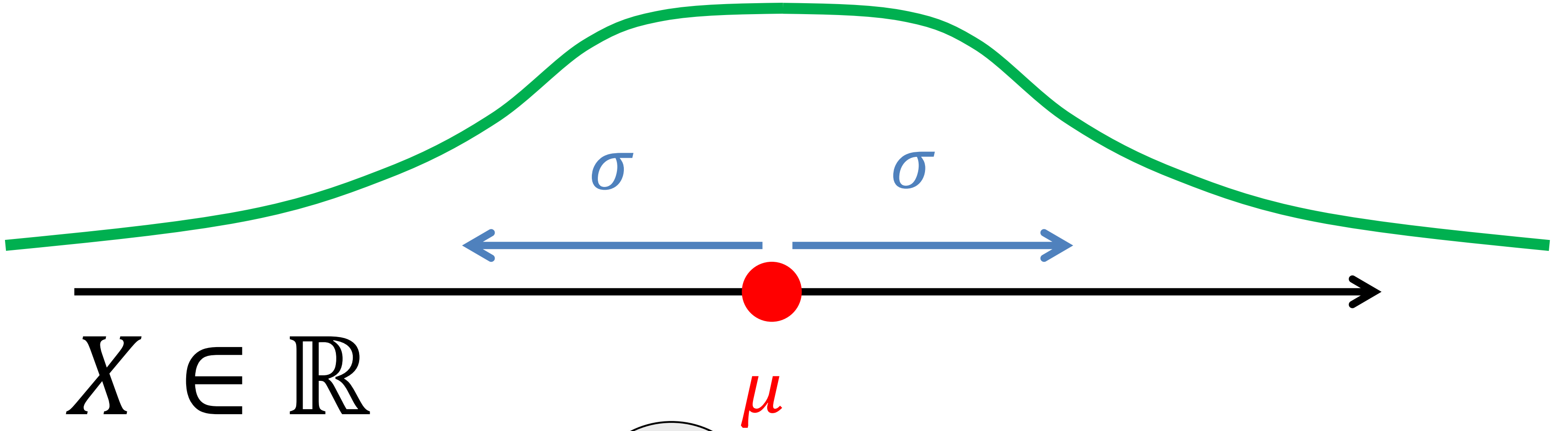


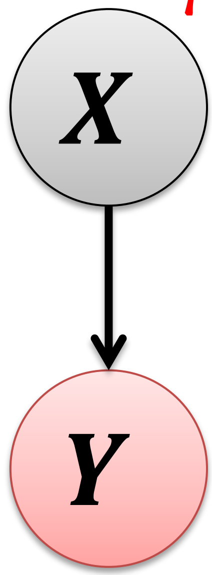
$X \in \mathbb{R}$



$\Pr[X = x] = \mathcal{N}(\mu, \sigma^2)$

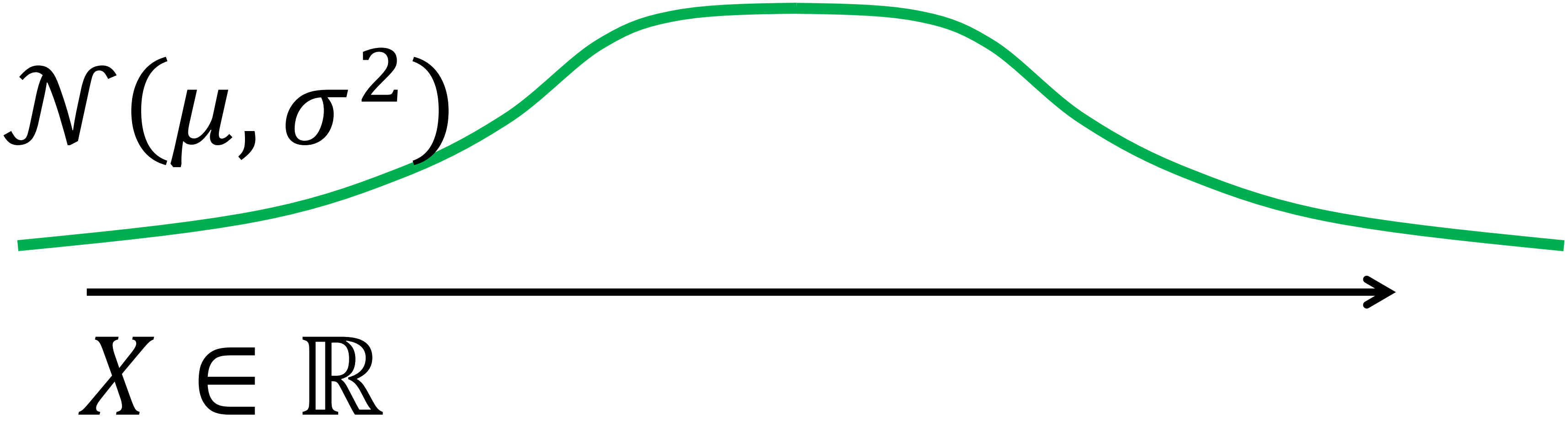
$\Pr[Y|x] = ?$

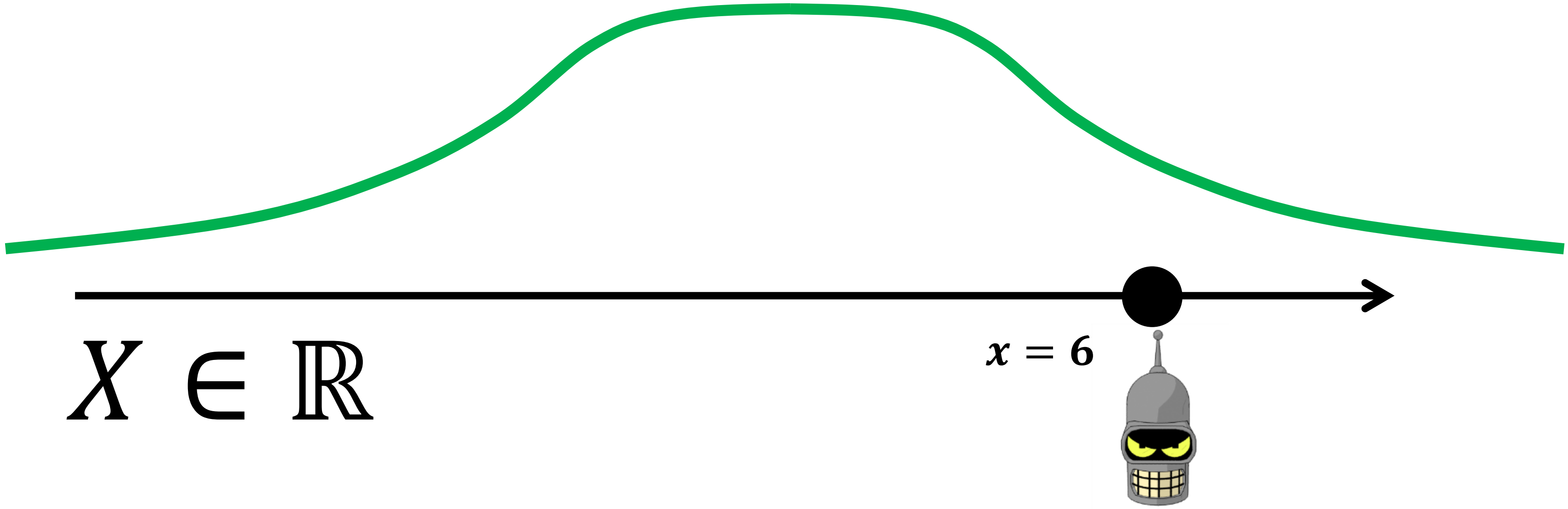


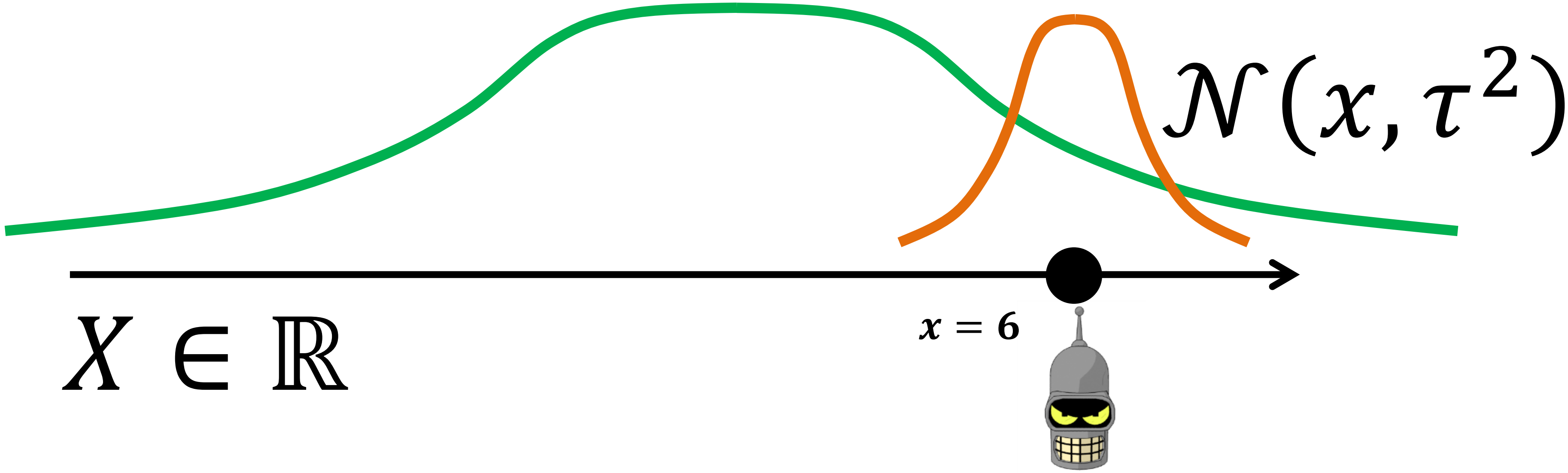


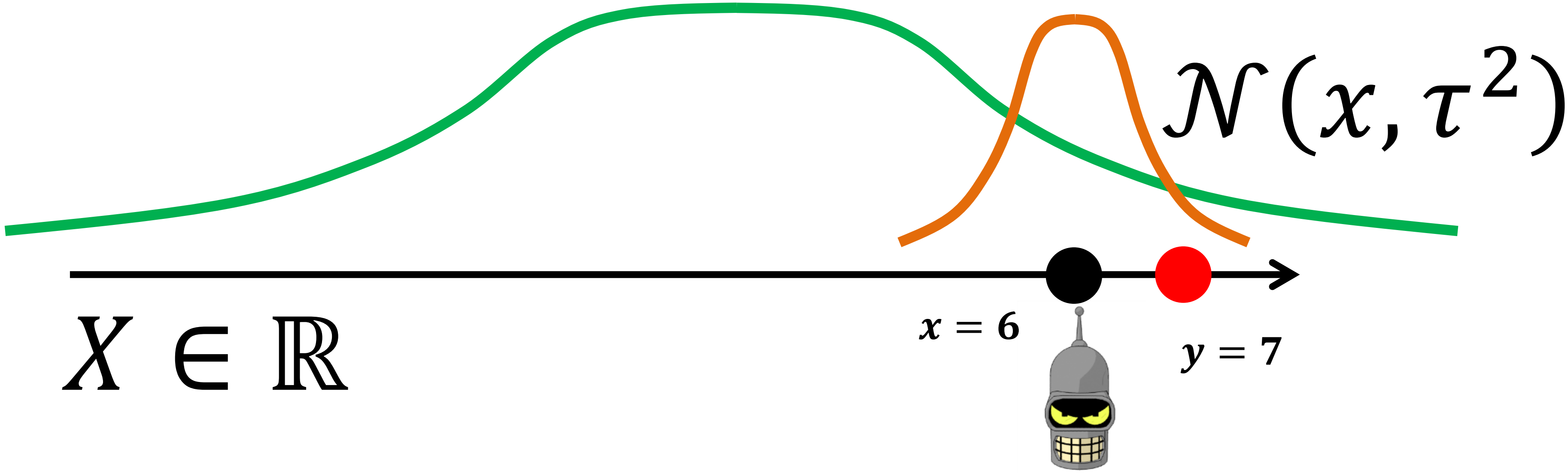
$$\Pr[X = x] = \mathcal{N}(\mu, \sigma^2)$$

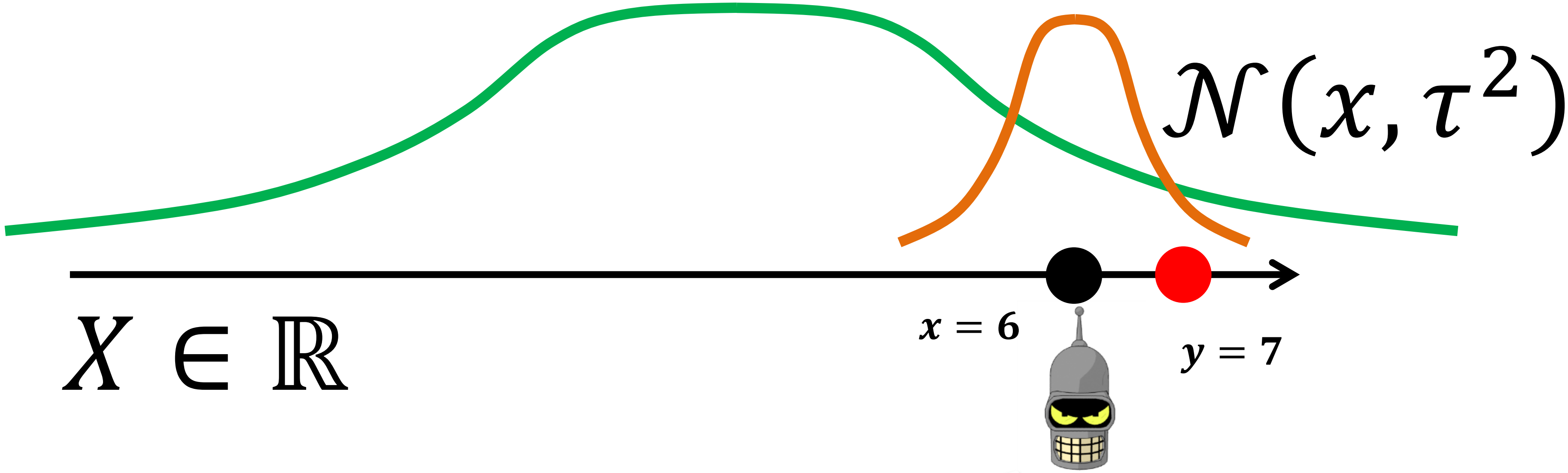
$$\Pr[Y|x] = \mathcal{N}(x, \tau^2)$$


$\mathcal{N}(\mu, \sigma^2)$
$X \in \mathbb{R}$

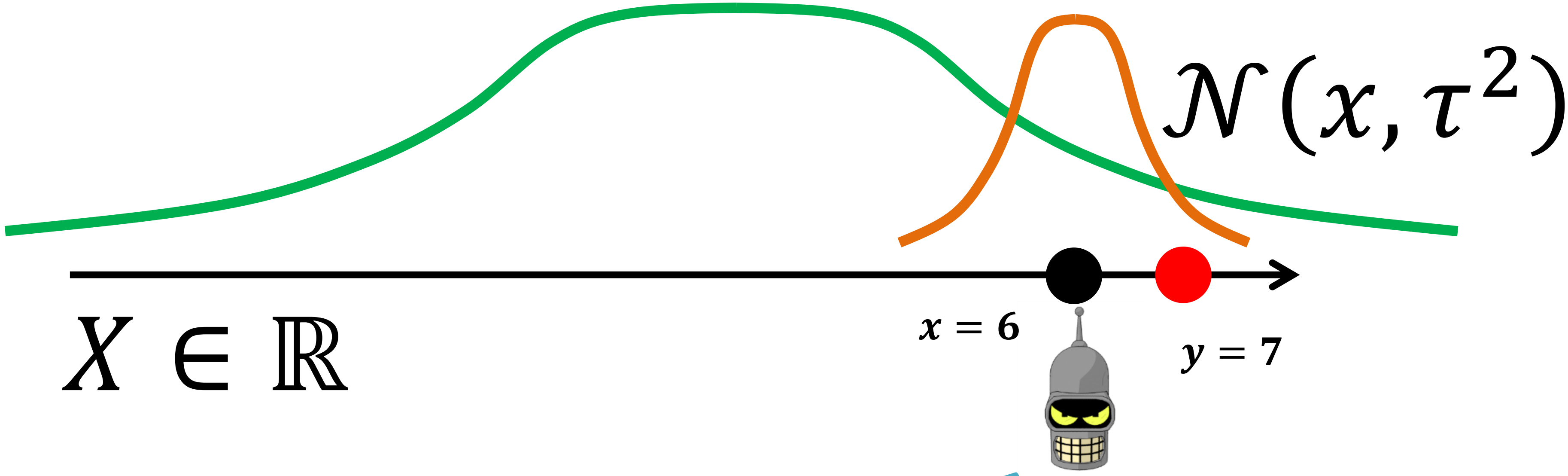

$x = 6$
$X \in \mathbb{R}$


$\mathcal{N}(x, \tau^2)$
$x = 6$
$X \in \mathbb{R}$

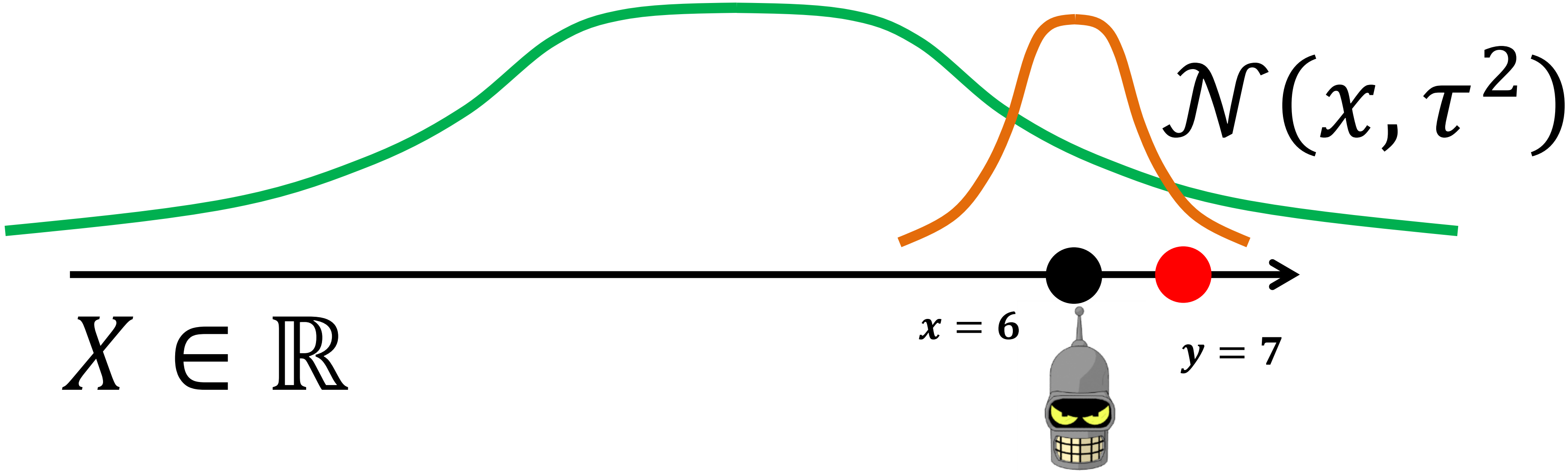

$\mathcal{N}(x,\tau^2)$
$X \in \mathbb{R}$
$x = 6$
$y = 7$



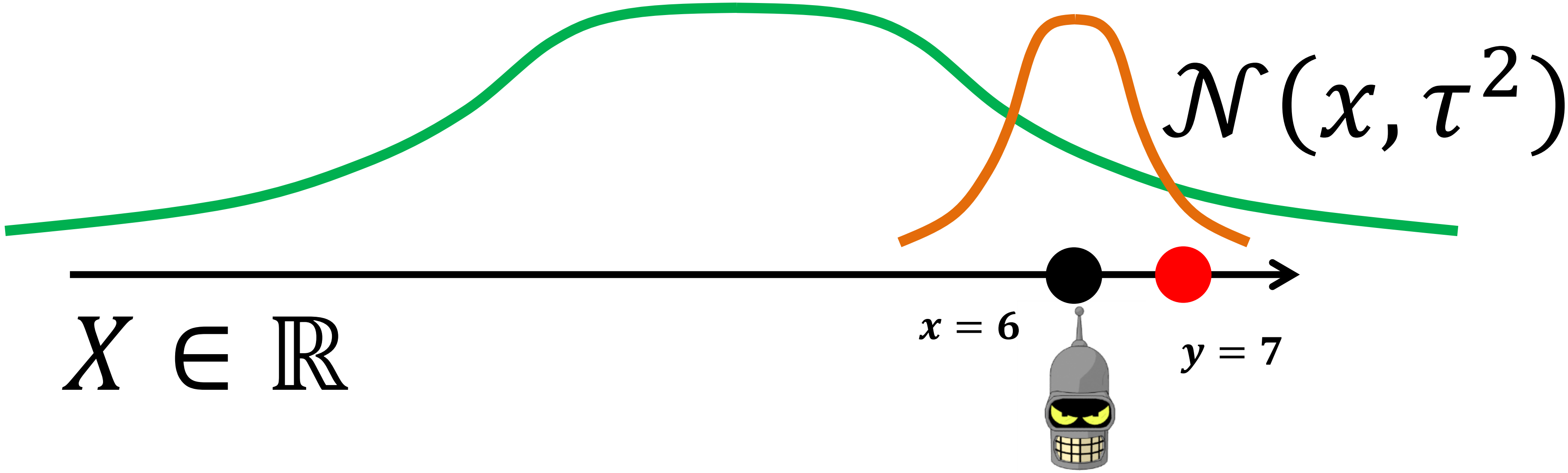
I observed my position at 7.
Where am I actually?



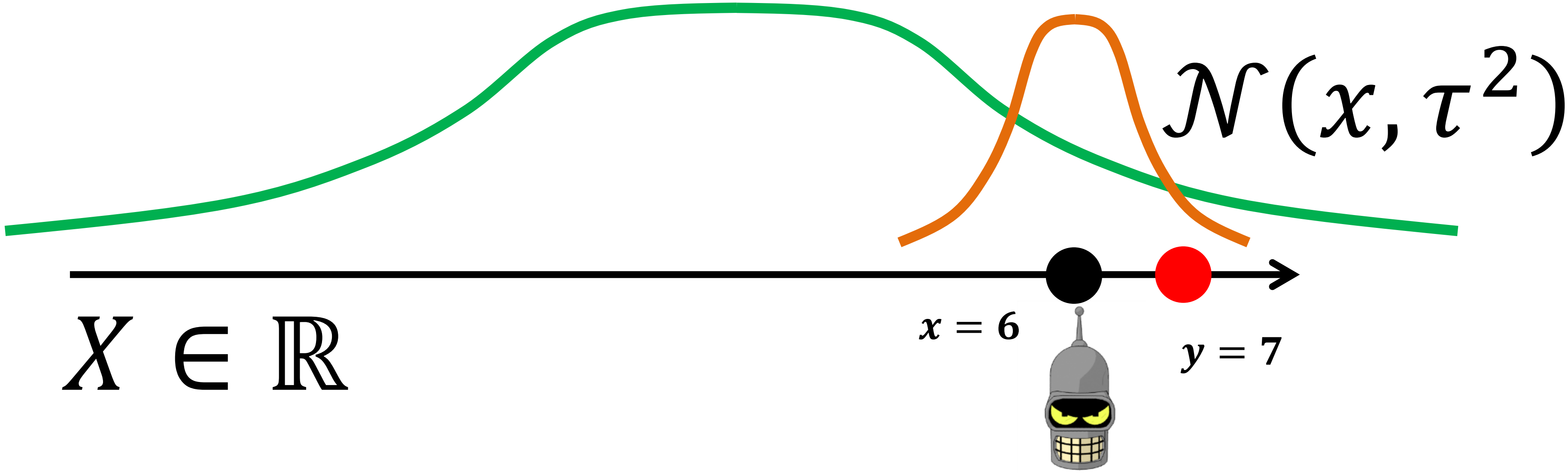
$$\Pr[x \mid y = 7] = ?$$



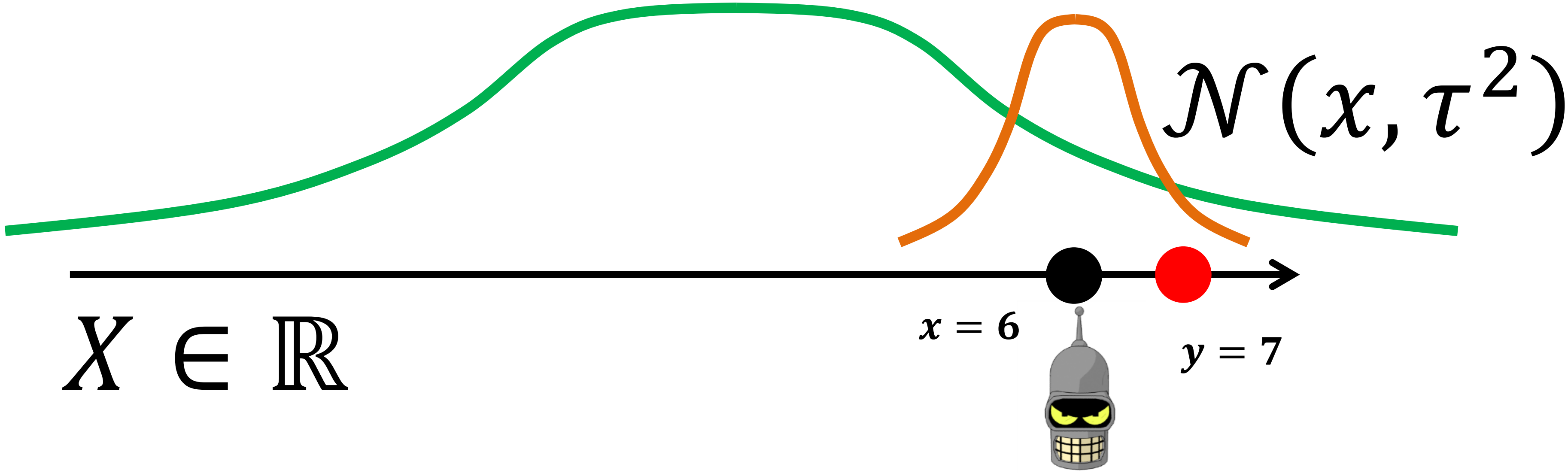
$$\Pr[x \mid y = 7] \propto \Pr[7 \mid x] \Pr[x]$$



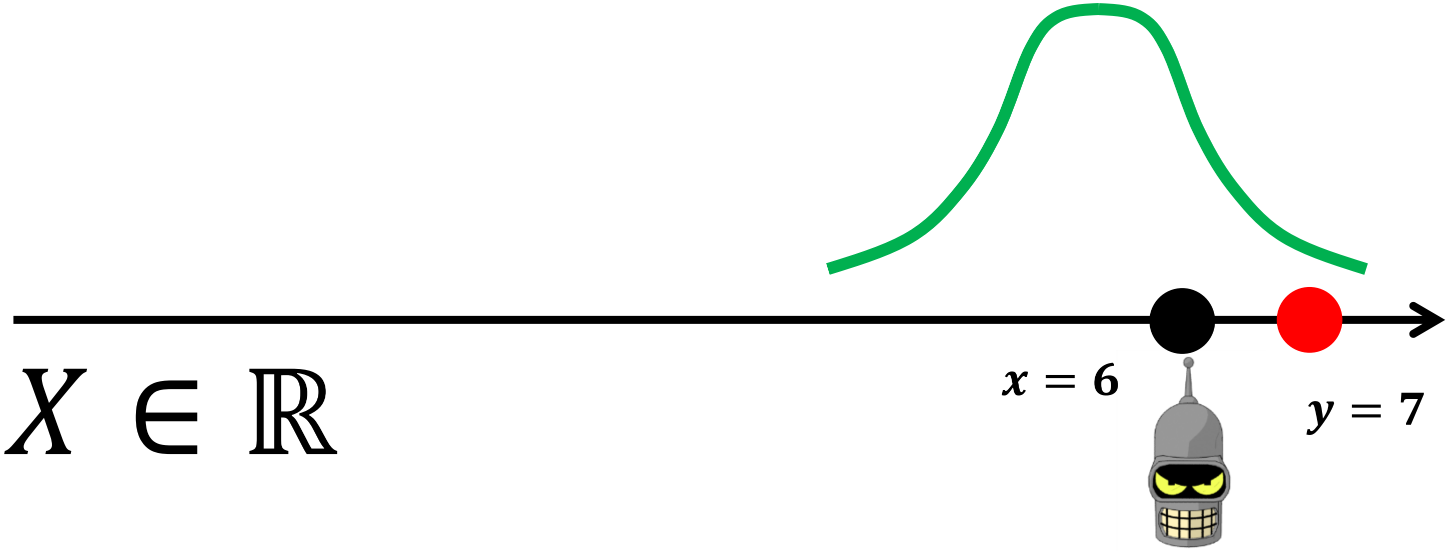
$$\Pr[x|y=7]$$
$$\propto \exp\left(-\frac{1}{2}\frac{(7-\mathrm{x})^2}{\tau^2}\right)\Pr[x]$$



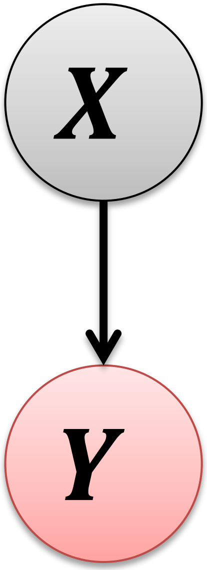
$$
\Pr[x|y = 7] \propto \exp\left(-\frac{1}{2}\frac{(7 - \mathrm{x})^2}{\tau^2}\right)\exp\left(-\frac{1}{2}\frac{(x - \mu)^2}{\sigma^2}\right)
$$



$$\Pr[x|y = 7] \propto \exp\left(-\frac{1}{2}\frac{(\mathrm{x} - \mu_{\mathrm{new}})^2}{\sigma_{\mathrm{new}}^2}\right)$$

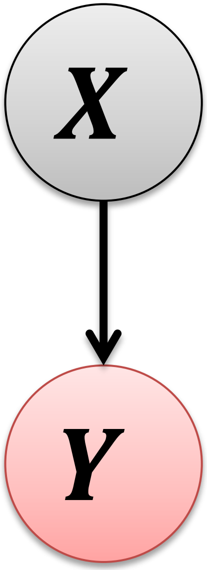


$$\Pr[x|y = 7] \propto \exp\left(-\frac{1}{2}\frac{(\mathrm{x} - \mu_{\mathrm{new}})^2}{\sigma_{\mathrm{new}}^2}\right)$$



$$\Pr[X = x] = \mathcal{N}(\mu, \sigma^2)$$

$$\Pr[Y|x] = \mathcal{N}(x, \tau^2)$$



$$\Pr[X = x] = \mathcal{N}(\mu, \sigma^2)$$

$$\Pr[Y|x] = \mathcal{N}(x, \tau^2)$$

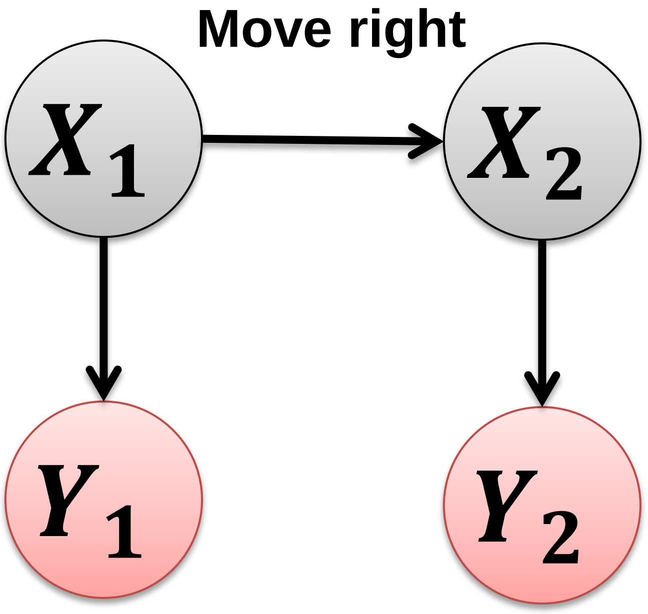
$$\Pr[X|y] = \mathcal{N}(\mu_{\text{new}}, \sigma_{\text{new}}^2)$$

$$\Pr[X|y] = \mathcal{N}(\mu_{\text{new}}, \sigma^2_{\text{new}})$$

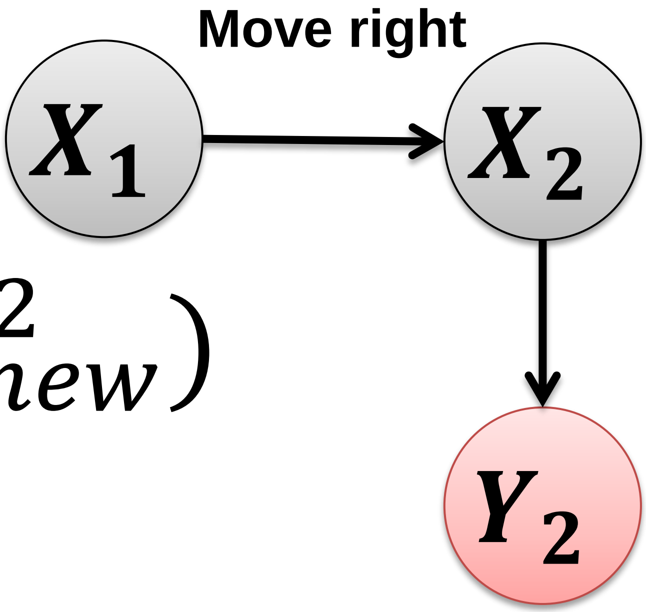
$$\mu_{new} = \frac{\mu\sigma^{-2} + y\tau^{-2}}{\sigma^{-2} + \tau^{-2}}$$

$$\sigma_{new} = \frac{\sigma^2\tau^2}{\sigma^2 + \tau^2}$$

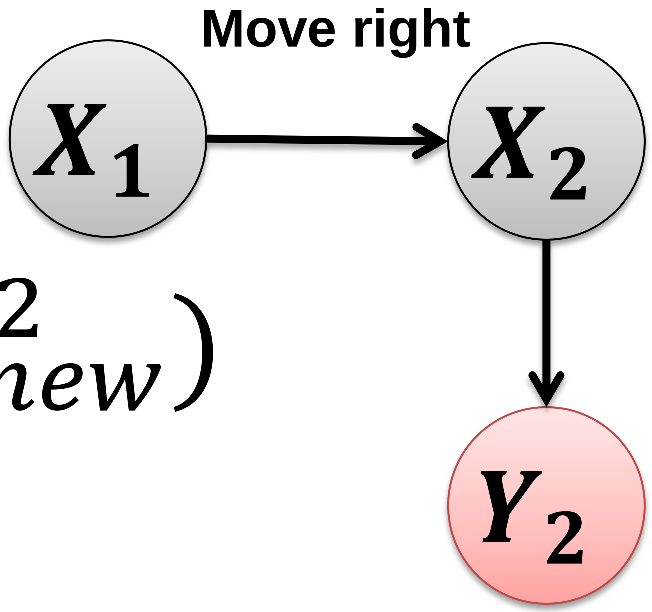
$\mathcal{N}(\mu, \sigma^2)$



$$\mathcal{N}(\mu_{new}, \sigma^2_{new})$$

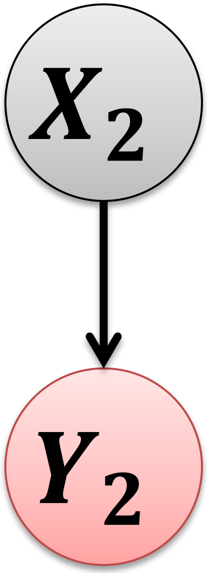


$$(\Delta\mu, \sigma_{move}^2)$$



$$\mathcal{N}(\mu_{new}, \sigma_{new}^2)$$

$$\mathcal{N}(\mu_{new} + \Delta\mu, \sigma_{new}^2 + \sigma_{move}^2)$$



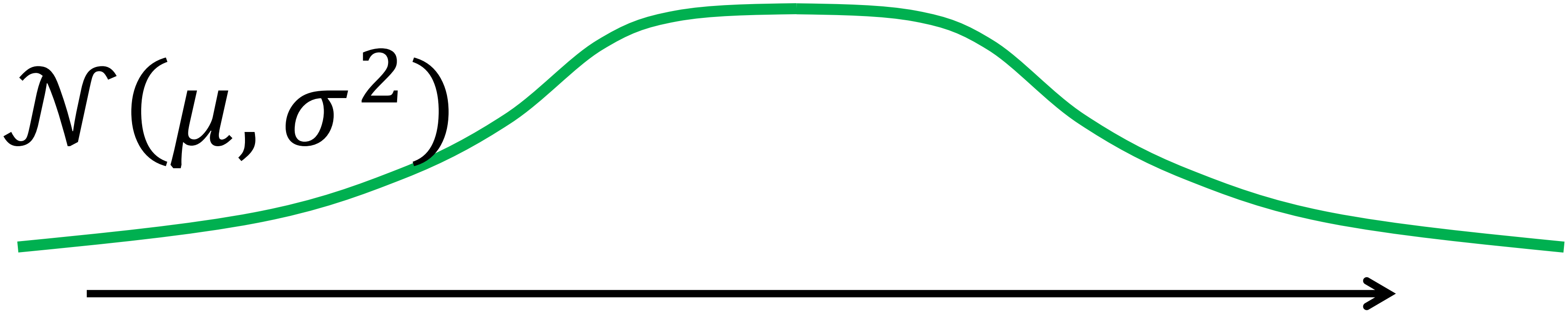
(One-dimensional) Kalman filtering

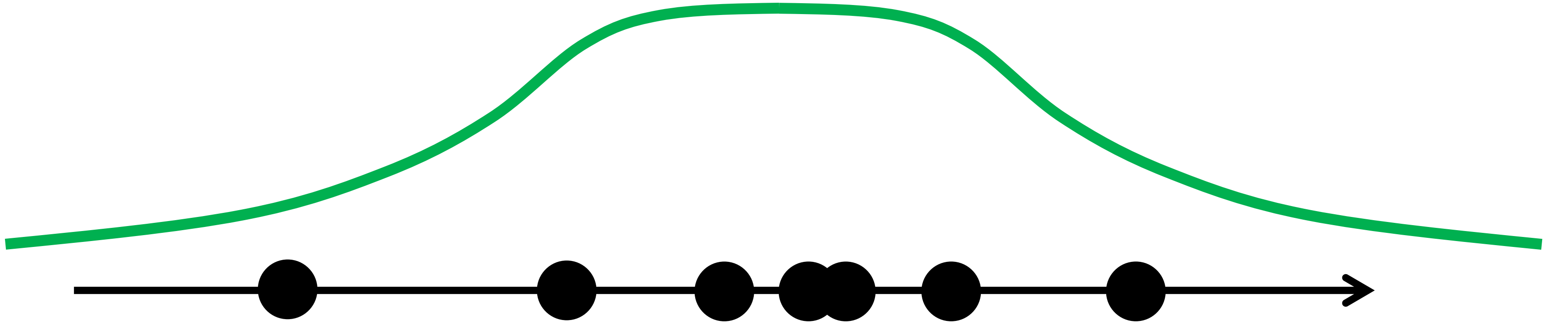
- Initialize (μ, σ^2)
- Repeat forever:
  - **Observe y, then**
    - $\mu := \frac{\sigma^{-2}\mu + \tau^{-2}y}{\sigma^{-2} + \tau^{-2}}$ $\quad \sigma^2 := \frac{\sigma^2\tau^2}{\sigma^2 + \tau^2}$
  - **Move**
    - $\mu := \mu + \Delta\mu$ $\quad \sigma^2 := \sigma^2 + \sigma_{move}^2$

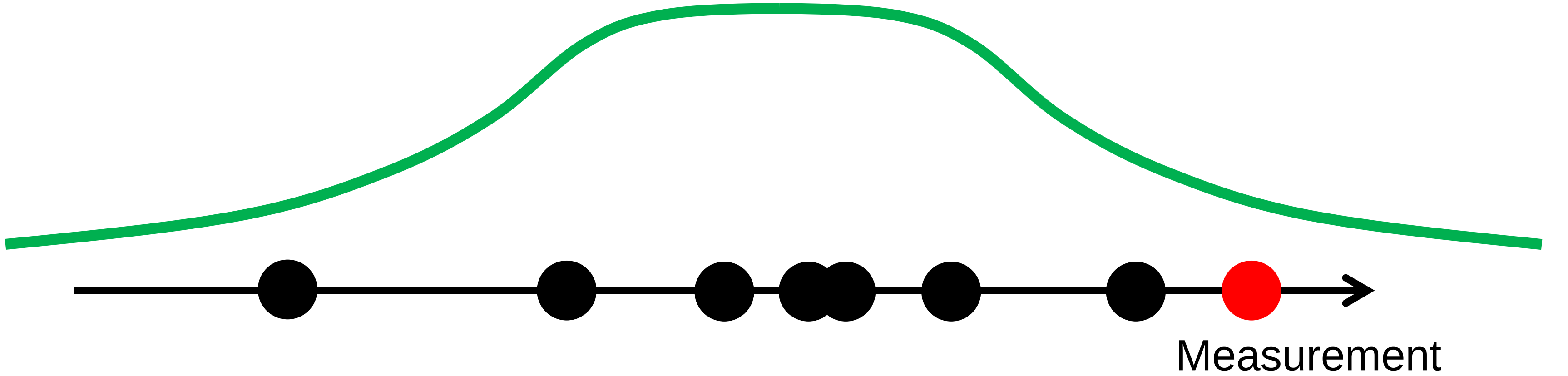
How to handle non-discrete distributions?

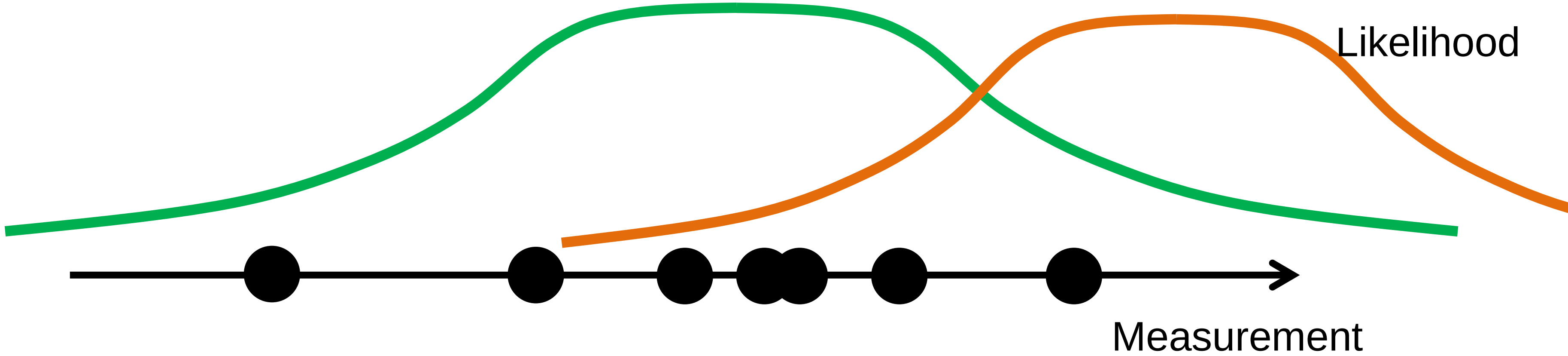
- **Parameterized function families**
- **Population-based representation**

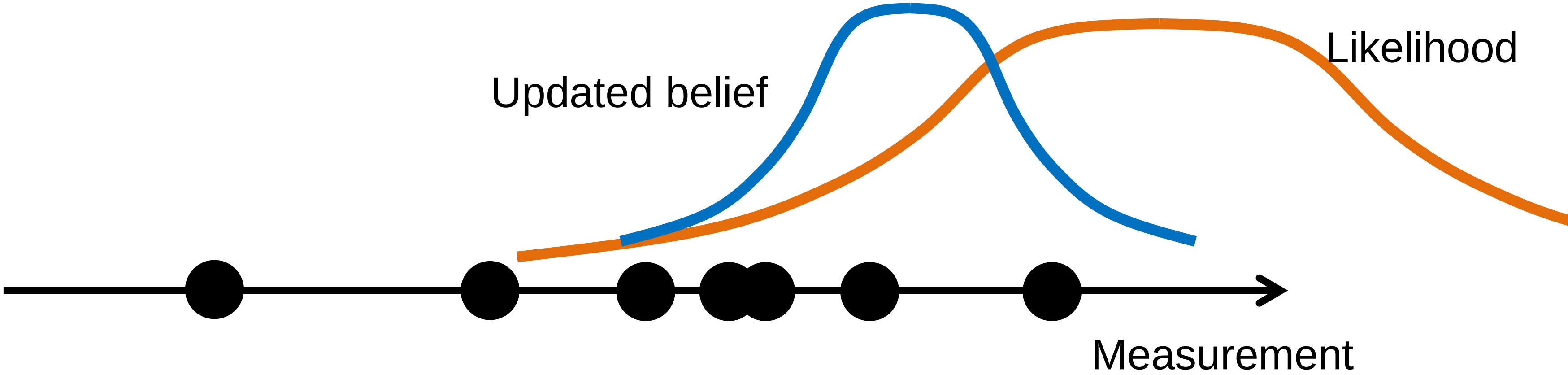
$$\mathcal{N}(\mu, \sigma^2)$$

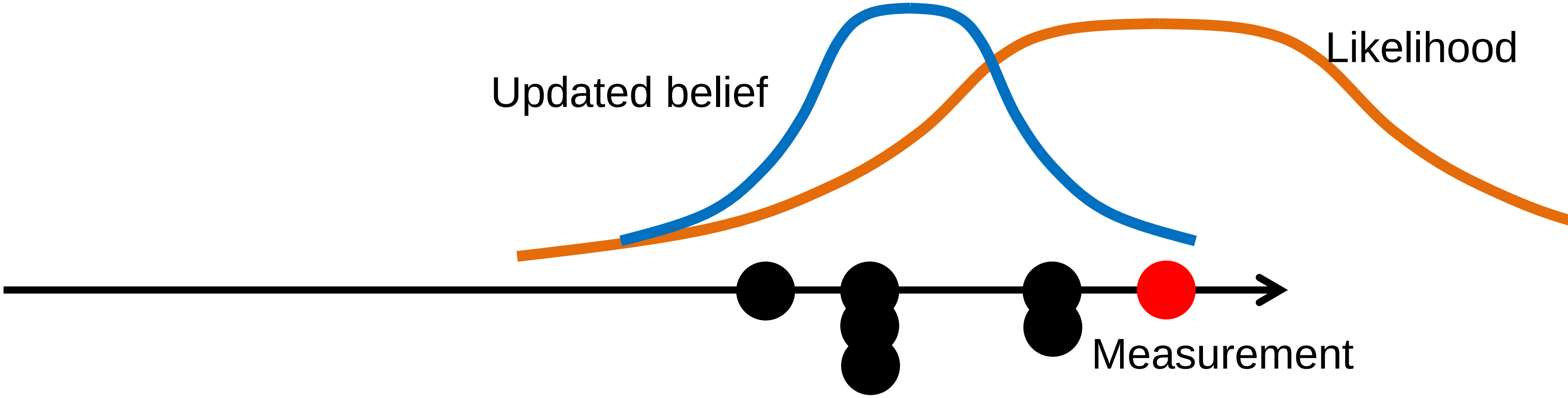





Measurement

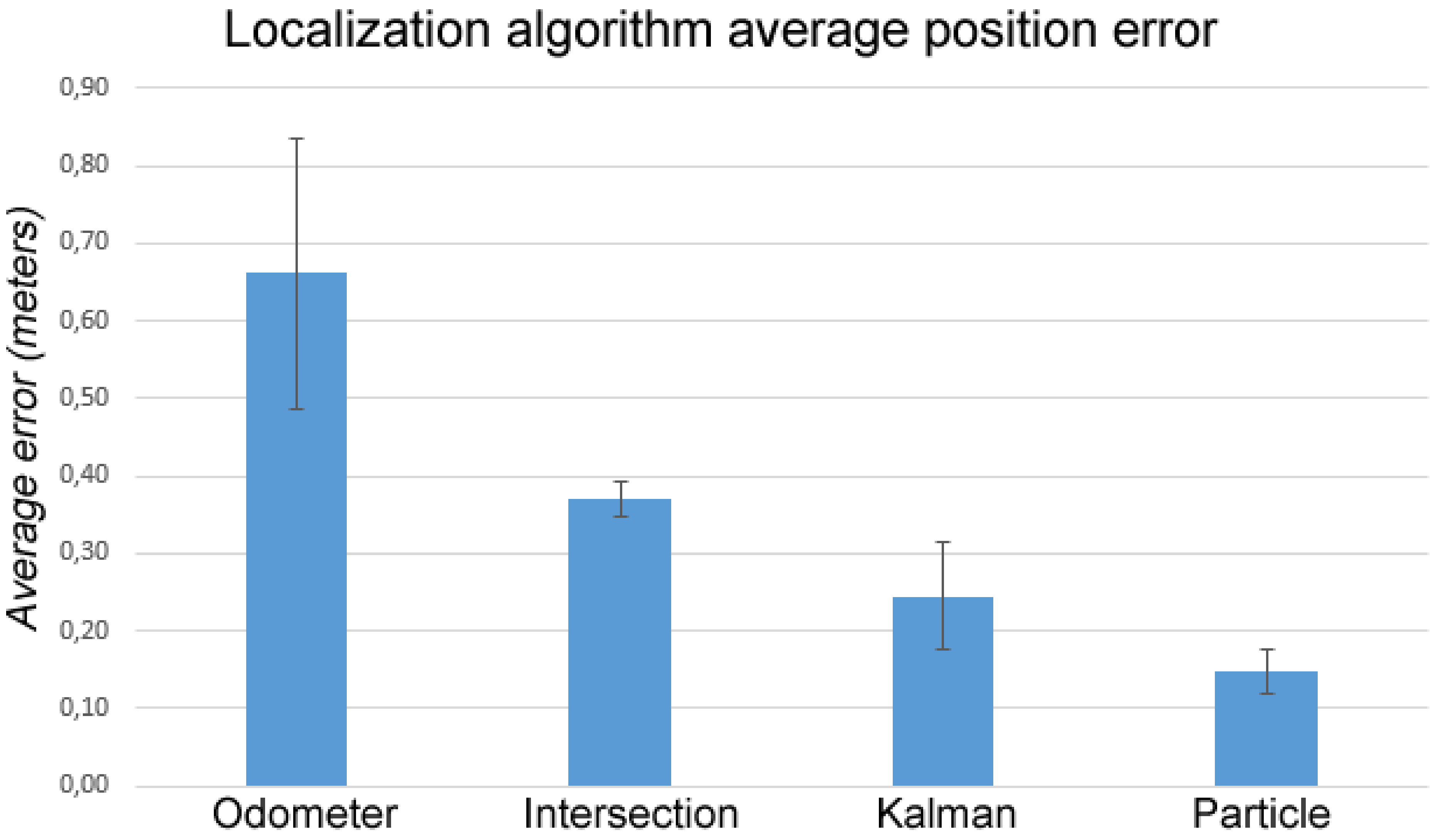

Likelihood
Measurement


Updated belief
Likelihood
Measurement

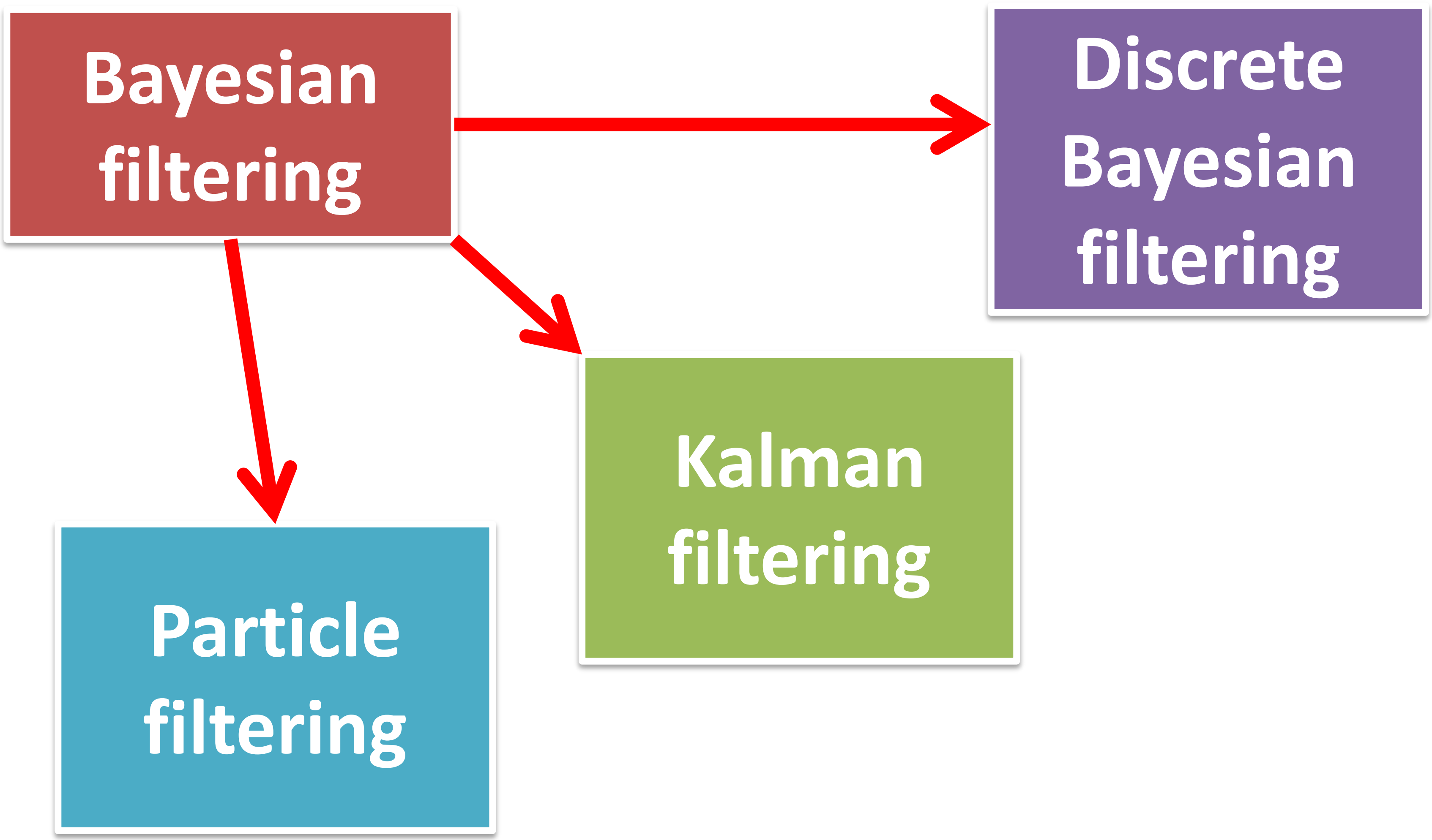

Updated belief
Likelihood
Measurement

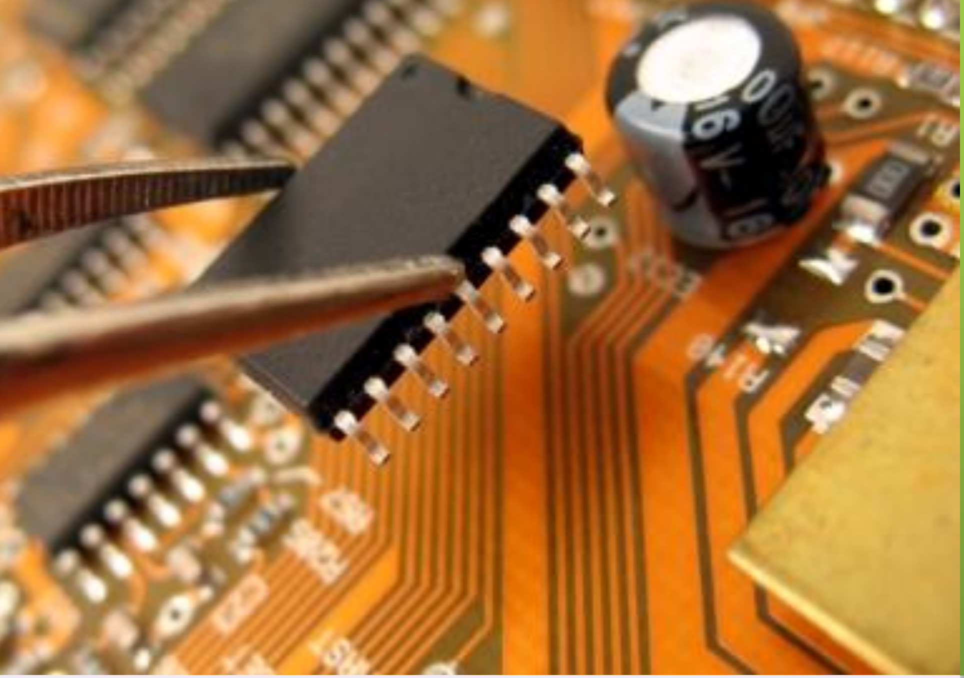
Particle filtering

- Initialize set of particles
- Repeat forever:
  - **Observe y,**
    - **Compute likelihood for each particle**
    - **Resample particles**
  - **Move**
    - **Add movement vector + noise to each particle**


Localization algorithm average position error
Average error (meters)
0,90
0,80
0,70
0,60
0,50
0,40
0,30
0,20
0,10
0,00
Odometer
Intersection
Kalman
Particle

Summary


Bayesian filtering
Discrete Bayesian filtering
Kalman filtering
Particle filtering



$$y = h * u$$

$$\hat{y}(z) = \hat{h}(z)\hat{u}(z)$$

$$y[i] = \alpha_0 u[i] + \alpha_1 u[i-1] +$$

$$x[t+1] = Ax[t] + Bu[t]$$
$$y[t] = Cx[t] + Du[t]$$

OpenCV